Some Further Properties of the Cumulative Offer Process

Mustafa Oğuz Afacan*

Abstract

In a matching with contracts framework, we investigate extension/resource monotonicity and respecting improvements properties of the cumulative offer process (COP). Extension monotonicity says that no doctor is to be better off whenever others start preferring more contracts to being unmatched. Resource monotonicity, on the other hand, requires that no doctor becomes worse off whenever hospitals start hiring more doctors. The COP becomes extension and resource monotonic whenever contracts are unilateral substitutes (US) satisfying an irrelevance of rejected contracts condition (IRC). This result, in particular, implies that the COP is population monotonic under US and the IRC, that is, no doctor is to be worse off whenever others leave the market. These findings, along with the stability of the COP, enable us to obtain opposite comparative statics results for the hospital side of the market. We then turn to the respecting improvements property, which states that no doctor should be harmed if some of his contracts become more popular. With an additional law of aggregate demand condition, we show that the COP respects improvements.

JEL classification: C78, D44, D47.

Keywords: extension monotonicity, resource monotonicity, population monotonicity, respecting improvements, the cumulative offer process, matching with contracts.

*Faculty of Arts and Social Sciences, Sabancı University, 34956, İstanbul, Turkey. E-mail: mafacan@sabanciuniv.edu

1 Introduction

Hatfield and Milgrom (2005) introduce a matching with contracts framework that admits both Kelso and Crawford (1982)'s labor market and the conventional matching (without contracts) models as its special cases.[1] They generalize the substitutes condition in the conventional matching literature (e.g., see Roth and Sotomayor (1990)) and introduce a "*Cumulative Offer Process*" (henceforth, COP), which is a generalization of Gale and Shapley (1962)'s doctor-proposing deferred acceptance algorithm (henceforth, DA). Hatfield and Milgrom (2005) show that the COP produces a stable allocation if contracts are substitutes. On the other hand, whenever hospitals do not have underlying preferences, generating their choices, Aygün and Sönmez (2013) obtain that result under an additional irrelevance of rejected contracts condition (henceforth, IRC).

Since then, the COP has constituted the main mechanism in the matching with contracts literature; therefore, it is important to understand more about the COP. To this end, the extant literature has already investigated some of its properties, and this paper takes one step further in that direction. Specifically, we consider three comparative statistics properties: *extension monotonicity*, *resource monotonicity* (e.g., see Chun and Thomson (1988)), and *respecting improvements* (e.g., see Balinski and Sönmez (1999)) and investigate under what kinds of contracts the COP satisfies them. The first two are solidarity requirements. Extension monotonicity states that no doctor is to be better off whenever some other doctors start preferring more contracts to being unmatched. The well-known population monotonicity property (e.g., see Thomson (1983)), which says that no doctor is to be worse off when some other doctors leave the market, is implied by extension monotonicity. Similarly, resource monotonicity says that no doctor is to lose whenever hospitals start hiring more. The last condition guarantees that no doctor is to receive a worse contract after his contracts become more popular.

[1]Echenique (2012) shows that the matching with contracts problem can be embedded into Kelso and Crawford (1982)'s setting under the substitutes condition of Hatfield and Milgrom (2005).

We first show that the COP is both extension and resource monotonic (hence, in particular, population monotonic) whenever contracts are *unilateral substitutes* (see Hatfield and Kojima (2010). Henceforth, US) satisfying the IRC. As corollaries of these two results, we find that hospitals benefit if doctors prefer more contracts to being unmatched and that whenever some hospitals start hiring more doctors, other hospitals do not gain. However, while we obtain these powerful comparative statics results under US and the IRC, the COP fails to respect improvements under them. A remedy for this negative result is the additional "Law of Aggregate Demand" (henceforth, LAD) condition of Hatfield and Milgrom (2005). The COP respects improvements under US, the LAD, and the IRC. We also show that the results are tight in the sense that neither of them holds if one drops the IRC or weakens US to the *bilateral substitutes* condition of Hatfield and Kojima (2010) (henceforth, BS), and that the respecting improvement result is lost without the LAD.

The extant literature already provides similar comparative statistics results for the COP in various settings. As formally discussed in the Model section, our results differ them in important ways. However, let us briefly mention here. First, our resource monotonicity notion is a new one with the property that it considers the largest class of choice expansions in the literature. The same kind of generality holds for our respecting improvement notion as well. Besides, we obtain these general comparative statistics results in a choice domain which is not included in any other choice domain in which similar comparative statistics results are obtained.

There is a recent surge on practical matching market studies in a matching with contract framework, including Sönmez and Switzer (2013), Sönmez (2013), and Aygün and Bo (2014). All of these papers propose the COP to be used in their respective practical matching problems, and their choice function proposals satisfy US, the LAD, and the IRC. Hence, all of our comparative statistics results hold in their settings. Therefore, beside its pure theoretical interest, our paper has a practical value.

2 Literature Review

The COP has received much attention from researchers. Hatfield and Kojima (2010) introduce BS, which is weaker than substitutability, and show that the COP is stable under BS. They also obtain that under the stronger US condition (still weaker than substitutability), the COP produces the doctor-optimal stable allocation, and it indeed collapses to the DA. With the additional LAD, they further show that the COP becomes strategy-proof (indeed group strategy-proof). Aygün and Sönmez (2012) obtain all of these results with the additional IRC for the case where hospitals do not have underlying preferences.

The extant literature provides several comparative statistics results for both the COP and its predecessor DA. In a many-to-many matching with contracts framework, Chambers and Yenmez (2014) consider path independent choice functions and show that under the COP, whenever an agent's choice function expands, then no other agent in the same side is better off while all the agents in the other side are at least weakly better off. This result has been generalized to the more general class of choice functions that have path independent completions by Yenmez (2015). Chambers and Yenmez (2014) also show that whenever firms merge and their unified choice functions expand (or consolidate), every worker is at least weakly better off (worse off) while the converse is true for every remaining firm.

Echenique and Yenmez (2015) consider the standard controlled school-choice without contracts model. They show that under substitutability and the IRC, whenever a school's choice function expands, then no student becomes worse off under the COP. They obtain similar results whenever choices are induced by various affirmative action policies. In a doctor-hospital matching with regional caps framework, Kamada and Kojima (2015) obtain that under substitutability, the LAD, and the IRC, whenever regions' choices expand, then each doctor is at least weakly better off under the COP.

In their many-to-one labor market model, Kelso and Crawford (1982) show that adding a worker (or a firm) benefits the firm (worker) side under the salary-adjustment process (Crawford and Knoer (1981)) under some assumptions including substitutes. Crawford (1991) extends their analysis to the many-to-many matching. In the conventional matching without contracts problem, Gale and Sotomayor (1985) show that the DA satisfies population monotonicity for responsive preferences. Kojima and Manea (2010) extend this result to the class of acceptant and substitutable preferences. In the standard house allocation (unit-capacity) setting, Ehlers and Klaus (2016) characterize the class of responsive-DA rules with strategy-proofness and population monotonicity as well as some other mild auxiliary axioms. Moreover, in the multi-capacity case, they obtain a characterization of the class of choice based-DA mechanisms by using resource monotonicity, strategy-proofness, and some other mild auxiliary axioms.

In a school-choice setting, Balinski and Sönmez (1999) is the first to introduce a respecting improvement notion. They show that the DA is the only stable mechanism that respects improvements. Sönmez and Switzer (2013) and Sönmez (2013) adapt that notion to the cadet-branch matching in a matching with contracts framework and show that the COP respects improvements under their choice function proposals. Kominers and Sönmez (2016) and Afacan (2016) obtain similar results in their respective matching with contracts setting.

3 The Model and Results

There are finite sets of doctors D, hospitals H, and contracts X. Each *contract* $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Each doctor can sign at most one contract. The *null contract*, denoted by $\emptyset$, represents being unmatched.

Each doctor $d \in D$ has a strict *preference relation* P_d over $\{x \in X: \ x_D = d\} \cup \{\emptyset\}$. Given any two contracts x', x where $x'_D = x_D = d$, we write $x' R_d x$ only if $x' P_d x$ or $x' = x$. A contract x is *acceptable* to doctor d if $x P_d \emptyset$. It is otherwise unacceptable. The chosen contract of doctor d from $X' \subseteq X$ is given as

$$C_d(X') = \max_{P_d}[\{x \in X': \ x_D = d\ \} \cup \{\emptyset\}].$$

Let $C_D(X') = \bigcup_{d\in D} C_d(X')$. The preference profile of doctors is $P = (P_d)_{d\in D}$, and we write $P_{D'} = (P_d)_{d\in D'}$ to denote the preferences of a group of doctors $D' \subset D$. For $X' \subseteq X$, let $X'_D = \{d \in D :\ \exists\, x \in X' \text{ with } x_D = d\}$.

Each hospital h has a *choice function* $C_h : 2^X \to 2^X$ defined as follows: For any $X' \subseteq X$, $C_h(X') \in \{X'' \subseteq X' :$ (for each $x \in X''$, $x_H = h$) and (for any $x', x'' \in X''$ such that $x' \neq x''$, $x'_D \neq x''_D$)$\}$.

Let $C_H(X') = \bigcup_{h\in H} C_h(X')$. The choice function profile of hospitals is $C = (C_h)_{h\in H}$, and for $H' \subset H$, we write $C_{H'} = (C_h)_{h\in H'}$ for that of hospitals in H'. As the set of doctors and hospitals will be fixed in the rest of the paper, we write (P, C) to denote the problem.

A set of contracts $X' \subseteq X$ is an *allocation* if $x, x' \in X'$ and $x \neq x'$ imply $x_D \neq x'_D$. We extend the preferences of doctors over the set of allocations in a natural way as follows: For any given two allocations X' and X'', $X' P_d X''$ if and only if $\{x' \in X' :\ x'_D = d\}\ P_d \{x'' \in X'' :\ x''_D = d\}$. For an allocation X' and hospital h, let $X'_h = \{x \in X' :\ x_H = h\}$.

A *mechanism* ψ is a function such that for any problem (P, C), it produces an allocation $\psi(P, C)$. Hatfield and Milgrom (2005) generalize Gale and Shapley (1962)'s celebrated DA to the current matching with contracts setting by introducing the following cumulative offer process (COP).

Step 1: One arbitrarily chosen doctor d offers her favorite acceptable contract x_1. The offer-receiving hospital h holds the contract if $x_1 = C_h(\{x_1\})$ and rejects it otherwise. Let $A_h(1) = \{x_1\}$ and $A_{h'}(1) = \emptyset$ for all $h' \neq h$.

In general,

Step t: One arbitrarily chosen doctor currently having no contract held by any hospital offers her preferred acceptable contract x_t from among those that have not been rejected in the previous steps. The offer-receiving hospital h holds x_t if $x_t \in C_h(A_h(t-1) \cup \{x_t\})$ and rejects it otherwise. Let $A_h(t) = A_h(t-1) \cup \{x_t\}$ and $A_{h'}(t) = A_{h'}(t-1)$ for all $h' \neq h$.

The algorithm terminates when every doctor has either a held contract by a hospital or all of his acceptable contracts rejected. As there are finitely many contracts, the algorithm terminates in some finite step T. The final outcome is $\bigcup_{h\in H} C_h(A_h(T))$.

The COP fails to produce an allocation without any structure on the hospital choices. In what follows, we give some well-known conditions that not only make the COP a well-defined mechanism, but also bring our comparative statistics results in the paper.

Definition 1. *Contracts satisfy the irrelevance of rejected contracts (IRC) for hospital h if for any $Y \subset X$ and $z \in X \setminus Y$,*

$$z \notin C_h(Y \cup \{z\}) \Rightarrow \; C_h(Y) = C_h(Y \cup \{z\}).$$

Definition 2. *Contracts are unilateral substitutes (US) for hospital h if there are no set of contracts $Y \subset X$ and $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}),\; z_D \notin Y_D, \text{ and } z \in C_h(Y \cup \{x, z\}).$$

Definition 3. *Contracts satisfy the law of aggregate demand (LAD) for hospital h if, for all $X' \subset X'' \subseteq X$, $|C_h(X')| \leq |C_h(X'')|$.*

In what follows, we present our comparative statistics properties and results.

Definition 4. *For a group of doctors $D' \subseteq D$, a preference profile $P'_{D'}$ of doctors D' is an extension of $P_{D'}$ if for any doctor $d \in D'$ and any pair of contracts $x, x' \in X$ such that $x_D = x'_D = d$, the followings hold,*

(i) if $x P_d x' P_d \emptyset$, then $x P'_d x' P'_d \emptyset$,

(ii) if $x P_d \emptyset P_d x'$, then $x P'_d x'$, and

(iii) $\exists\; x'' \in X$ with $x''_D = d$ such that $\emptyset P_d x''$ and $x'' P'_d \emptyset$.

In words, a preference list is an extension of another preference list if the former contains more acceptable contracts while preserving the rankings[2] of the contracts that are acceptable with respect to the latter.

[2]If a contract is the top k^{th} alternative, then its ranking is k.

Definition 5. *A mechanism ψ is extension monotonic (under some conditions) if for any problem (P, C) (in which the contracts satisfy these conditions) and any group of doctors $D' \subset D$ with any extension $P'_{D'}$ of $P_{D'}$, $\psi(P, C) R_d \psi(P_{D \setminus D'}, P'_{D'}, C)$ for any $d \in D \setminus D'$.*

Extension monotonicity is a solidarity requirement that no doctor is to be better off whenever some other doctors extend their preferences. A special case of extension occurs whenever some doctors initially find every contract unacceptable but then extend their preferences. In this case, extension monotonicity coincides with the well-known "population monotonicity" condition. Formally, let $P_d^{\emptyset}$ denote the preferences of doctor d where any contract is unacceptable, and for any set of doctors $D' \subseteq D$, let $P_{D'}^{\emptyset} = (P_d^{\emptyset})_{d \in D'}$.

Definition 6. *A mechanism ψ is population monotonic (under some conditions) if for any problem (P, C) (in which the contracts satisfy these conditions) and any group of doctors $D' \subset D$, $\psi(P_{D \setminus D'}, P_{D'}^{\emptyset}, C) R_d \psi(P, C)$ for any $d \in D \setminus D'$.*

Population monotonicity says that no doctor is to be worse off whenever some other doctors leave the market by declaring that every contract is unacceptable. The following remark is a direct consequence of the above definitions.

Remark 1. *If a mechanism is extension monotonic (under some conditions), then it is population monotonic (under the same conditions). The converse is not true.*

Chambers and Yenmez (2014) and Echenique and Yenmez (2015) say that a choice function profile $C' = (C'_h)_{h \in H}$ is a [contract-wise] expansion of C if for any hospital $h \in H$ and set of contracts $X' \subseteq X$, $C_h(X') \subseteq C'_h(X')$. We now define a weaker version of this as follows. A choice function profile $C' = (C'_h)_{h \in H}$ is a *D-expansion* (doctor-wise expansion) of C if for any hospital $h \in H$ and set of contracts $X' \subseteq X$, $[C_h(X')]_D \subseteq [C'_h(X')]_D$. In words, the latter considers expansions where from any contracts set, hospitals do not reject any doctor that they previously choose. The former, on the other hand, considers more stringent expansions where from any contracts set, hospitals do not reject any contract that they previously select. It is easy to see that any (contract-wise) expansion is a D-expansion, though the converse is not true. D-expansion is a new notion, and it considers the largest class of choice expansions in the literature.[3]

Definition 7. *A mechanism ψ is resource monotonic (under some conditions) if for any problem (P, C) and any D-expansion C' of C (in which the contracts satisfy these conditions under both C and C'), $\psi(P, C')R_d\psi(P, C)$ for every $d \in D$.*

Resource monotonicity is another solidarity condition requiring that no doctor is worse off whenever hospitals start hiring more doctors.

Theorem 1.

(i) The COP is extension monotonic under US and the IRC; hence in particular, it is population monotonic under US and the IRC.

(ii) The COP is resource monotonic under US and the IRC.

Proof. See the Appendix A. □

While doctors lose from other doctors extending their preferences and benefit from hospitals hiring more doctors, the converse is true for the hospital side of the market, as stated in the following corollary.

Corollary 1. *For any problem (P, C) in which the contracts are US satisfying the IRC under C,*

(i) if for any group of doctors $D' \subseteq D$, $P'_{D'}$ is an extension of $P_{D'}$, $COP(P, C) = X'$, and $COP(P_{D \setminus D'}, P'_{D'}, C) = X''$, then for any hospital h, $C_h(X' \cup X'') = X''_h$.

(ii) Let C' be a D-expansion of C such that the contracts are US satisfying the IRC under C'. If $COP(P, C) = X'$ and $COP(P, C') = X''$, then for any hospital $h \in \{h \in H : C_h = C'_h\}$, $C_h(X' \cup X'') = X'_h$.

Proof. See the Appendix A. □

[3] Both Kamada and Kojima (2015) and Yenmez (2015) consider contract-wise expansions. In the standard matching without contract setting, D-expansion coincides with contract-wise expansion.

Remark 2. In the Appendix B, we show that Theorem 1 is tight in the sense that if one replaces US with BS or drops the IRC, then the result no longer holds even under the additional LAD.

Remark 3. In a many-to-many matching with contracts framework, Chambers and Yenmez (2014) obtain similar comparative results under path independence. Yenmez (2015) then generalizes Chambers and Yenmez (2014)'s results to the class of choice functions that admit path independent completions. Our and their results are independent of each other. First of all, as already pointed out, our resource monotonicity condition is more general than theirs. On the other hand, US and the IRC together is weaker than path independence.[4] And we currently do not know the relation between US together with the IRC and path independent completion.[5] Therefore, the respective choice domains of these papers are different as well. Hence, there is no logical relation between the resource monotonicity results. On the other hand, Chambers and Yenmez (2014) consider a larger class of expansions for the doctor-side. However, again due to their stronger path independence assumption, their result does not imply ours.[6]

Another important property is that doctors should not be penalized whenever their contracts become more popular among hospitals. This property is called "respecting improvements" in the literature.

Below, we introduce a broad improvement notion and show that the COP respects improvements under US, the LAD, and the IRC. Our analysis is more general than both Sönmez and Switzer (2013) and Sönmez (2013) in two ways. First, their improvement notions are choice-rule specific. More specifically, they both refer to a rise on the branches' (or hospitals' in our terminology) ranking lists as improvement, which makes all the contracts

[4]Aizerman and Malishevski (1981) show that path independence is equivalent to substitutability and the IRC.

[5]Kadam (2015) shows that US and the IRC together implies path independent completion for the case where hospitals have underlying preferences. As the presence of preferences brings some other restrictions as well as the IRC (see Aygün and Sönmez (2012)), it is not clear whether or not Kadam (2015)'s result carries over to the current paper's setting where hospitals have choices as primitive.

[6]Yenmez (2015) does not consider expansions in the doctor-side.

of the associated cadet (doctor) more popular under their specific choice functions. In contrast, our improvement formulation is broadly defined for any choice function regardless of the presence of a ranking list. Moreover, we show our result for any hospital choice function satisfying US, the LAD, and the IRC. Sönmez and Switzer (2013) and Sönmez (2013), however, conduct their respective analyses just for the specific choice functions (so-called "USMA" and "ROTC" choices). As these choice functions satisfy US, the LAD, and the IRC, our result implies theirs; however, the converse is not true. Similarly, Kominers and Sönmez (2016) conduct a narrower respecting improvement analysis for their specific choice functions.

Definition 8. *We say that $C' = (C'_h)_{h \in H}$ is an improvement over $C = (C_h)_{h \in H}$ for doctor d if, for any hospital h and set of contracts $X' \subseteq X$,*

(i) if $x \in C_h(X')$ where $x_D = d$, then $x' \in C'_h(X')$ for some $x' \in X'$ where $x'_D = d$;

(ii) if $d \notin [C_h(X') \cup C'_h(X')]_D$, then $C_h(X') = C'_h(X')$.

In words, the first condition states that if a contract of doctor d is chosen from any given contracts set under C, then some contract (not necessarily the same one) of the doctor continues to be chosen under C'. In particular, if he has only one contract in the given set and it is chosen, then the same contract is to be chosen under C'. Hence, we can say that the popularity of every single contract of doctor d relative to those of the other doctors at least weakly increases (it may strictly increase whenever his not chosen contract under C starts to be chosen under C'). On the other hand, it allows the popularity of doctor d's contracts to change within themselves. For instance, doctor d's specialist contract might become more popular than his generalist contract. In this case, the former may be chosen under C' even though the latter is chosen under C (whenever they are both available in the given set of contracts).[7] The second condition, on the other hand, states that if doctor d is not chosen from a given contracts set under both C and C' (that is, no contract of his is chosen), then

[7] At this point, one might wonder what would happen if doctor d prefers his generalist contract to the specialist one. As doctors make offers in decreasing order of their preferences in the course of the COP, it would not be a problem for him under US, the LAD, and the IRC, as will be shown in Theorem 2.

the chosen contracts do not change.

Definition 9. *Mechanism ψ respects improvements (under some conditions) if for any problem (P, C) and C' such that C' is an improvement over C for doctor d (and the contracts satisfy these conditions under both C and C'), $\psi(P, C') R_d \psi(P, C)$.*

Theorem 2. *The COP respects improvements under US, the LAD, and the IRC.*

Proof. See the Appendix C. □

Remark 4. In the Appendix D, we show that Theorem 2 is tight in the sense that the result would no longer hold if we omit the LAD or the IRC or replace US with BS.

4 Conclusion

The matching with contracts framework of Hatfield and Milgrom (2005) is very rich in that the traditional matching problem (without contracts) and the labor market model of Kelso and Crawford (1982) are special cases of it. Recent studies demonstrate the practical importance of the matching with contracts formulation as well. Therefore, understanding more about the COP, which is the main mechanism in the matching with contract literature, is of both theoretical and practical interest. To this end, this paper studies three comparative statistics properties of the COP. We show that the COP is both extension and resource monotonic under US and the IRC, and with the additional LAD, it respects improvements as well. Investigating further properties of the COP might be a fruitful direction for future research.

Appendix

A Proof of Theorem 1 and Corollary 1

Before proceeding to the proofs, let us first define stability (Hatfield and Milgrom (2005)). An allocation X' is stable if

(1) $C_D(X') = C_H(X') = X'$ and

(2) there exist no hospital h and set of contracts $X'' \neq C_h(X')$ such that $X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'')$.

From Hatfield and Kojima (2010) and Aygün and Sönmez (2012), we know that the COP produces stable allocations under BS (weaker than US) and the IRC.

Proof of Theorem 1. In the proof, we assume that in the course of the COP, doctors having no held contract simultaneously offer their respective best offer among the ones that have not been rejected previously. This version is called simultaneous-offer COP. This supposition is legitimate as Hirata and Kasuya (2014) show that under BS and the IRC, the simultaneous-offer COP coincides with the sequential-offer COP (with any order in which doctors are to make an offer) described in the Model Section, and we assume stronger US and the IRC.

(i). We first show that the COP is extension monotonic under US and the IRC. Let us now consider two different preference profiles of doctors P and P' where $P_d = P'_d$ for every doctor $d \neq d'$, and $P'_{d'}$ is an extension of $P_{d'}$. The proof of the general case where more than one doctor extends preferences follows from the iterated application of this case.

In order to show that no doctor $d \in D \setminus d'$ is better off at P' under the COP, we will prove that any rejected contract x at P continues to be rejected at P'. This will in turn imply that no doctor $d \in D \setminus d'$ benefits at P'.

Let x be a contract that is rejected in the first step of the COP at P. Let Y_h^k and $Y_h'^k$ be the set of offers hospital h receives in step k at P and P', respectively.

By the definitions of P and P', we have $Y_h^1 \subseteq Y_h'^1$ for any hospital $h \in H$. Moreover,

$[Y_h'^1 \setminus Y_h^1]_D \cap [Y_h^1]_D = \emptyset$. If $x_H = h$, then it implies that $x \in Y_h^1$, $x_D \notin [Y_h^1 \setminus \{x\}]_D$ (that is, doctor x_D does not have contract other than x in Y_h^1), and $x \notin C_h(Y_h^1)$. Then, as $Y_h^1 \subseteq Y_h'^1$ and $[Y_h'^1 \setminus Y_h^1]_D \cap [Y_h^1]_D = \emptyset$, by US, we also have $x \notin C_h(Y_h'^1)$ (note that $x \in Y_h^1$ and $x_D \notin [Y_h^1 \setminus \{x\}]_D$).

Above shows that in the course of the COP, any contract x that is rejected in the first step at P continues to be rejected in the first step at P' as well. This implies that $Y_h^2 \subseteq Y_h'^2$ for every hospital $h \in H$.

Let us now consider any contract x that is rejected in the second step of the COP at P. We now show that it is rejected within the first two steps of the COP at P'. Let A_h^k and $A_h'^k$ be the set of offers held by hospital h by step k at P and P', respectively. Let $x_H = h$. By our supposition, $x \notin C_h(Y_h^1 \cup Y_h^2)$. Under US and IRC, we know that in any step of the COP, no hospital renegotiates for any contract it rejected in the previous steps (Hatfield and Kojima (2010) and Aygün and Sönmez (2012). In other words, the COP coincides with the DA). This implies that $C_h(Y_h^1 \cup Y_h^2) = C_h(A_h^1 \cup Y_h^2)$. As x is rejected in the second step, we have $x \in A_h^1 \cup Y_h^2$ and $x \notin C_h(A_h^1 \cup Y_h^2)$.

As the same as above, we have $C_h(Y_h'^1 \cup Y_h'^2) = C_h(A_h'^1 \cup Y_h'^2)$. As $Y_h^1 \cup Y_h^2 \subseteq Y_h'^1 \cup Y_h'^2$, $x \in Y_h'^1 \cup Y_h'^2$. Assume for a contradiction that $x \in C_h(Y_h'^1 \cup Y_h'^2)$, hence $x \in C_h(A_h'^1 \cup Y_h'^2)$ (because $C_h(Y_h'^1 \cup Y_h'^2) = C_h(A_h'^1 \cup Y_h'^2)$ under US and IRC, as pointed out above). Let us now define $S = A_h'^1 \setminus A_h^1$. As $A_h^1 \cup S \subseteq Y_h'^1$ (note that $A_h'^1 \subseteq A_h^1 \cup S$) and $C_h(Y_h'^1 \cup Y_h'^2) = C_h(A_h'^1 \cup Y_h'^2)$, by the IRC, we have $C_h(A_h'^1 \cup Y_h'^2) = C_h((A_h^1 \cup S) \cup Y_h'^2)$. Hence in particular, $x \in C_h((A_h^1 \cup S) \cup Y_h'^2)$. For ease of notation, let $Z = (A_h^1 \cup S) \cup Y_h'^2$. Note that $A_h^1 \cup Y_h^2 \subseteq Z$. Moreover, by construction $x \in A_h^1 \cup Y_h^2$ and there is no other contract of x_D in $A_h^1 \cup Y_h^2$ (as x is rejected in the second step of the COP at P). We now define $Z' = Z \setminus \{x' \in Z : \ x'_D = x_D$ and $x' \neq x\}$. Since the only contract doctor x_D has in $A_h^1 \cup Y_h^2$ is x, we have $A_h^1 \cup Y_h^2 \subseteq Z'$.

We have $x \in C_h(Z)$, which implies that any other contract of doctor x_D in Z is rejected (if any). Therefore, by the IRC, $C_h(Z) = C_h(Z')$. We now have $x \notin C_h(A_h^1 \cup Y_h^2)$, $x \in C_h(Z')$ where $A_h^1 \cup Y_h^2 \subseteq Z'$, and x is the only contract of doctor x_D in Z'. This contradicts US, showing that $x \notin C_h(A_h^{\prime 1} \cup Y_h^{\prime 2})$ as well.

We therefore prove that any contract rejected in the second step of the COP at P is rejected within the first two steps at P'. This implies that $Y_h^3 \subseteq Y_h^{\prime 3}$ for any hospital $h \in H$. By following the same arguments above, we can show that any contract that is rejected in the third step of the COP at P is also rejected within the first three steps at P' as well. Continuing in the same manner would eventually prove that in the course of the COP, any rejected contract at P is rejected at P' as well. This in turn shows that no doctor is better off whenever some other doctors extend their preferences, which means that the COP is extension monotonic under US and the IRC.

(ii). In this second part of the proof, we show that the COP is resource monotonic under US and the IRC. To this end, let us consider a problem instance (P, C) and a D-expansion C' where (i) the contracts are US satisfying the IRC under both C and C', and (ii) $C'_h = C_h$ for any $h \in H \setminus \{h'\}$. The proof of the general case where more than one hospital's choices expand follows from the iterated application of this case. In order to conclude that $COP(P, C')R_dCOP(P, C)$ for every doctor $d \in D$, we will show that any rejected contract in the course of the COP at C' is rejected at C as well. This will in turn imply the result.

Similar to above, let Y_h^k and $Y_h^{\prime k}$ be the set of offers that hospital h receives in step k at C and C', respectively. By construction, we have $Y_h^1 = Y_h^{\prime 1}$ for every hospital $h \in H$. Let x be a contract that is rejected in the first step of the COP at C'. Then, by the definition of C and C', and the fact that the only contract that doctor x_D has in $Y_h^{\prime 1}(= Y_h^1)$ is x, contract x is rejected in the first step of the COP at C as well. This implies that $Y_h^{\prime 2} \subseteq Y_h^2$ for every hospital $h \in H$.

Let x be a contract that is rejected in the second step of the COP at C'. Then, as the same as above, we write A_h^k and $A_h^{\prime k}$ for the set of offers held by hospital h by step k at C and C', respectively. As x is rejected in the second step, we have $x \in A_h^{\prime 1} \cup Y_h^{\prime 2}$, $x \notin C'_h(A_h^{\prime 1} \cup Y_h^{\prime 2})$, and there is no other contract of doctor x_D in $A_h^{\prime 1} \cup Y_h^{\prime 2}$.

Let us first consider $x_H = h'$. From above analysis, we know that $Y^1_{h'} = Y'^1_{h'}$ and $Y'^2_{h'} \subseteq Y^2_{h'}$. Therefore, we have $A'^1_{h'} \cup Y'^2_{h'} \subseteq Y^1_{h'} \cup Y^2_{h'}$. If $x \notin C_{h'}(Y^1_{h'} \cup Y^2_{h'})$, then there is nothing to prove. Let us suppose that $x \in C_{h'}(Y^1_{h'} \cup Y^2_{h'})$. We next define $Z = [Y^1_{h'} \cup Y^2_{h'}] \setminus \{x' \in Y^1_{h'} \cup Y^2_{h'} : x'_D = x_D \text{ and } x' \neq x\}$. By the IRC, $C_{h'}(Y^1_{h'} \cup Y^2_{h'}) = C_{h'}(Z)$; hence in particular, $x \in C_{h'}(Z)$. On the other hand, as x is the only contract of doctor x_D in $A'^1_{h'} \cup Y'^2_{h'}$ and $A'^1_{h'} \cup Y'^2_{h'} \subseteq Y^1_{h'} \cup Y^2_{h'}$, we also have $A'^1_{h'} \cup Y'^2_{h'} \subseteq Z$. As $x \notin C'_{h'}(A'^1_{h'} \cup Y'^2_{h'})$, by US, we have $x \notin C'_{h'}(Z)$ as well. On the other hand, because $C'_{h'}$ is a D-expansion of $C_{h'}$, and the only contract that doctor x_D has in Z is x, $x \notin C'_{h'}(Z)$ implies that $x \notin C_{h'}(Z)$. This, however, contradicts our previous finding that $x \in C_{h'}(Z)$, showing that $x \notin C_{h'}(Y^1_{h'} \cup Y^2_{h'})$.[8]

The other case of $x_H = h \neq h'$ exactly follows from the same arguments above, except $C_h = C'_{h'}$. Once we define the same set Z as above, then we would have $x \notin C'_h(Z)$. Then, since $C'_h = C_h$, we have $x \notin C_h(Z)$, implying that $x \notin C_h(Y^1_h \cup Y^2_h)$.

We hence show that any contract that is rejected in the second step of the COP at C' is rejected within the first two steps of the COP at C. This implies that $Y'^3_h \subseteq Y^3_h$ for every hospital h. The same arguments above would show that any contract x that is rejected in the third step of the COP at C' is rejected within the first three steps at C as well. Then, continuing in the same manner would show that any rejected contract at C' is also rejected at C as well. This in turn implies that no doctor is worse off whenever hospitals start hiring more under US and the IRC. That is, the COP is resource monotonic under US and the IRC, which finishes the proof.

□

Proof of Corollary 1. (i). Assume for a contradiction that there exists a hospital h such that $C_h(X' \cup X'') \neq X''_h$. For ease of notation, let $C_h(X' \cup X'') = B$. First, observe that B is not a proper subset of X'_h. This is because, otherwise, we would have $C_h(X') = B$ (by the IRC). However, as X' is a stable allocation at (P, C), it would contradict with the fact that $C_h(X') = X'_h$. By the same arguments, B is not a proper subset of X''_h.

[8] I am especially grateful to Orhan Aygün for suggesting this way of proving.

Let $B = (B \setminus X''_h) \cup (B \cap X''_h)$. By our supposition, $B \setminus X''_h \neq \emptyset$. Note that $B \setminus X''_h \subseteq X'_h$. Let $P' = (P_{D \setminus D'}, P'_{D'})$. For any doctor d, let x'_d and x''_d be the contract that doctor d signs at allocations X' and X'', respectively. Due to the definition of extension and the fact that the COP coincides with the DA under US and the IRC (Hatfield and Kojima (2010) and Aygün and Sönmez (2012)), for any $d \in D'$ such that $(x'_d)_H = h$, we have $x'_d R_d x''_d$ and $x'_d P_d \emptyset$. This, along with the definition of $P'_{D'}$, in turn implies that for any $d \in D'$ such that $(x'_d)_H = h$, we also have $x'_d R'_d x''_d$ and $x'_d P'_d \emptyset$. On the other hand, from Theorem 1, we know that for any doctor $d \in D \setminus D'$ such that $(x'_d)_H = h$, $x'_d R_d x''_d$. Moreover, we have $B \setminus X''_h \neq \emptyset$. All of these findings in turn imply that at (P', C), X'' cannot be stable because doctors in $(B \setminus X''_h)_D$ and hospital h would rather sign their associated contracts at X'. This contradiction finishes the proof.

(ii). The proof of this part directly follows from Theorem 1 and the stability of the COP.

□

B The Tightness of Theorem 1

In some parts of the Appendix B and the Appendix D, we assume that hospitals have underlying preferences, inducing their choices. Therefore, in the following, we first give the formal definition of the induced hospital choice functions. Let $\succ_h$ be the hospital h's strict preferences over the set of subsets of contracts involving itself, then its choice function C_h is given as follows: For any $X' \subseteq X$, $C_h(X') = \max\limits_{\succ_h}\{X'' \subseteq X' : (\text{for each } x \in X'', x_H = h) \text{ and } (\text{for any } x', x'' \in X'' \text{ such that } x' \neq x'', x'_D \neq x''_D)\}$.

Definition 10 (Hatfield and Kojima (2010))**.** *Contracts are bilateral substitutes (BS) for hospital h if there are no set of contracts $Y \subset X$ and $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}),\ z_D, x_D \notin Y_D,\ \text{and } z \in C_h(Y \cup \{x, z\}).$$

In what follows, in the first two parts, we show that whenever US is weakened to BS, the COP loses its extension and resource monotonicity properties, even under the additional LAD. Then, the last two parts show that the same is true without the IRC.

(i) First, let us show that the COP is not population monotonic under BS, the IRC, and the LAD. Due to Remark 1, this implies that the COP is not extension monotonic under them either. Let us consider a problem in which $D = \{d_1, d_2, d_3\}$ and $H = \{h_1, h_2\}$. Let the hospitals have underlying preferences and consider the following preference profile:

$$\succ_{h_1}\colon \; \{x', z'\} \succ \{x, k\} \succ \{z, k\} \succ \{x', k\} \succ \{z', k\} \succ \{x, z\} \succ \{x, z'\} \succ z \succ x' \succ z' \succ k \succ \{x', z\} \succ x \succ \emptyset.$$

$$\succ_{h_2}\colon \; k' \succ \emptyset.$$

$$P_{d_1} : \; x', x, \emptyset; \; P_{d_2} : \; z, z', \emptyset; \; P_{d_3} : \emptyset, k, k'.$$

The contracts are such that $x'_H = x_H = z_H = z'_H = k_H = h_1$ and $k'_H = h_2$. It is easy to verify that the contracts under the hospital choice functions that are generated by the above preferences are BS (not US though, as shown below) satisfying both the IRC, and the LAD.[9] The COP outcome is $\{x, z\}$.[10] Let us now consider the different preferences of doctor d_3: $P'_{d_3} : \; k, \emptyset, k'$. In this case, COP outcome is $\{x', z'\}$; hence, doctor d_1 is better off, while doctor d_2 is worse off.[11] The COP is therefore not population monotonic under BS, the IRC, and the LAD.

(ii) For resource monotonicity, consider the same problem as above, except with different preferences of doctor d_3: $P''_{d_3} : \; k', k, \emptyset$, and of hospital h_2: $\succ'_{h_2}\colon \; \emptyset, k'$. Let us write C' for the hospital choices generated by $(\succ_{h_1}, \succ'_{h_2})$. Then, it is easy to verify that the contracts (with respect to C') are BS satisfying both the LAD and the IRC. The COP outcome

[9] As the choices are generated by the preferences, contracts automatically satisfy the IRC.

[10] To see this, let doctors make offers simultaneously (recall that the COP outcome is independent of the order in which doctors make offers and coincides with the simultaneous-offer version under BS and the IRC (Hirata and Kasuya (2014))). In the first step, contracts x' and z are offered by doctor d_1 and doctor d_2, respectively. The former is rejected, and then doctor d_1 offers contract x in the second step. Hence, the final outcome $\{x, z\}$ is obtained.

[11] In the first step, contracts x', z, and k are offered by doctor d_1, d_2, and d_3, respectively. Contract x' is rejected, and then doctor d_1 offers contract x in the second step. Among all these offered contracts, hospital h_1 now chooses $\{x, k\}$ and rejects contract z of doctor d_2. In the next step, doctor d_2 offers z', and the final outcome $\{x', z'\}$ is obtained.

at C' is $\{x', z'\}$. Let us now consider the preferences $\succ''_{h_2}$: $k', \emptyset$. If we write C'' for the choices generated by $(\succ_{h_1}, \succ''_{h_2})$, then C'' is a D-expansion of C' (indeed it is a (contract-wise) expansion of C'); moreover, the contracts continue to be BS satisfying both the LAD and the IRC (under C''). In this case, the COP outcome is $\{x, z, k'\}$; hence, doctors d_2 and d_3 both become better off, whereas doctor d_1 becomes worse off. The COP is therefore not resource monotonic under BS, the LAD, and the IRC either.

Note that in the above problems, even though the contracts are BS satisfying both the LAD and the IRC, they are not US: For instance, $C_{h_1}(\{x', z\}) = \{z\}$ and $C_{h_1}(\{x', z, z'\}) = \{x', z'\}$, violating US.

(iii) Let us consider one hospital h and two doctors, d_1, d_2. The doctors' preferences are as follows:

$$P_{d_1}: \ x, x', x'', \emptyset; \ P_{d_2}: \ \emptyset, y.$$

Hospital h has the following choice function C_h:

$C_h(x) = \emptyset$	$C_h(x', y) = \{y\}$	$C_h(x, x', y) = \{y\}$
$C_h(x') = \{x'\}$	$C_h(x, x') = \{x'\}$	$C_h(x, x'', y) = \{y\}$
$C_h(x'') = \{x''\}$	$C_h(x, x'') = \{x''\}$	$C_h(x', x'', y) = \{y\}$
$C_h(y) = \{y\}$	$C_h(x', x'') = \{x'\}$	$C_h(x, x', x'') = \{x'\}$
$C_h(x, y) = \{y\}$	$C_h(x'', y) = \{y\}$	$C_h(x, x', x'', y) = \{x\}$

It is easy to verify that the contracts under C_h are US satisfying the LAD (the IRC is not satisfied though, as shown below). The COP outcome is $\{x'\}$. Let us consider $P'_{d_2}: y, \emptyset$ and write $P' = (P_{d_1}, P'_{d_2})$. At P', the COP outcome is $\{x\}$, benefiting doctor d_1 from doctor d_2's extension. Therefore, the COP is not population monotonic, hence in particular not extension monotonic, under US and the LAD.

(iv). Let us consider the same hospital and doctors along with their preferences P' in the above part. Let C'_h be the choice function such that $C'_h(x, x', x'', y) = \{x'\}$ and it coincides with C_h above at all other choice sets. By definition, C'_h is a D-expansion of C_h. The COP outcome at (P', C'_h) is $\{x'\}$, hurting doctor d_1. Hence, without the IRC, the COP is not resource monotonic under US and the LAD.

Note that the IRC is violated in both part (iii) and (iv). For instance, $x'' \notin C_h(x, x', x'', y)$ but $C_h(x, x', x'', y) \neq C_h(x, x', y)$ (the same is true under C'_h as well).

C The Proof of Theorem 2

Proof of Theorem 2. Consider a problem (P, C) and C', which is an improvement over C for doctor d such that the contracts are US satisfying the LAD and the IRC under both C and C'. Let x and y be doctor d's signed contracts under the COP outcome at C and C', respectively. Assume for a contradiction that xP_dy (due to the stability of the COP, $yR_d\emptyset$, with the possibility that y can be the null-contract).

Let us now consider a false preference relation of doctor d: $P'_d : x, \emptyset$. It is easy to see that $COP(P, C)$ is stable at problem $(P'_d, P_{D\setminus d}, C')$. As $COP(P'_d, P_{D\setminus d}, C')R'_dCOP(P, C)$ (due to the COP outcome being the doctor-optimal stable allocation under US and the IRC (Hatfield and Kojima (2010) and Aygün and Sönmez (2012))), doctor d has an incentive to report P'_d at problem (P, C'). This, however, contradicts the strategy-proofness of the COP under US, the IRC, and the LAD (see Hatfield and Kojima (2010) and Aygün and Sönmez (2012)). Therefore, the COP respects improvements under US, the LAD, and the IRC.

□

D The Tightness of Theorem 2

(i). First, we demonstrate that the result does not hold under US and the IRC without the LAD. Let us consider a problem in which $D = \{d_1, d_2, d_3, d_4\}$ and $H = \{h_1, h_2, h_3\}$. Consider the following preferences of the doctors and the hospital.

$$\succ_{h_1}:\ z\ \succ \{y,k\} \succ ..(\textit{any pair of contracts involving } h_1).. \succ x \succ z \succ y \succ k \succ \emptyset,$$

$$\succ_{h_2}:\ y' \succ z' \succ x' \succ \emptyset,$$

$$\succ_{h_3}:\ k' \succ x'' \succ \emptyset,$$

$$P_{d_1}:\ x, x', x'', \emptyset;\ P_{d_2}:\ y, y', \emptyset;\ P_{d_3}:\ k, k', \emptyset,\ P_{d_4}:\ z', z, \emptyset.$$

The contracts are such that $x_H = y_H = k_H = z_H = h_1$, $y'_H = z'_H = x'_H = h_2$, and $k'_H = x''_H = h_3$. It is easy to verify that the contracts are US satisfying the IRC under the hospital choice function profile C that is generated by the above preferences. The COP outcome is $\{y, k, z', x''\}$. Let us now consider the different preferences of hospital h_1, generating its choices C'_{h_1}:

$$\succ'_{h_1}:\ z\ \succ x \succ \{y,k\} \succ ..(\textit{any pair of contracts with the same order as } \succ_{h_1}).. \succ z \succ y \succ k \succ \emptyset.$$

Let us write $C' = (C'_{h_1}, C_{-h_1})$.[12] It is easy to verify that the contracts are US satisfying the IRC under C', and C' is an improvement over C for doctor d_1. The COP outcome at C' is $\{z, y', k'\}$, making doctor d_1 worse off. Hence, the COP does not respect improvements under US and the IRC. Note that the LAD is violated at the above instance.[13]

(ii). Let us now show that the COP does not respect improvements under BS, the IRC, and the LAD. Consider only one hospital h and three doctors, d_1, d_2, and d_3. Let the preferences be as follows:

$$\succ_h:\ \{z', y\} \succ \{x', z\} \succ \{y, z\}.. \succ ..\textit{any pair of contracts}..\succ\ \textit{any singleton}..\succ \emptyset.$$

$$P_{d_1}:\ x, x', \emptyset;\ P_{d_2}:\ y, \emptyset;\ P_{d_3}:\ z, z', \emptyset.$$

Let C_h be the choice function generated by $\succ_h$. Then, $COP(P, C_h) = \{x', z\}$. Let us now consider C'_h generated by the following $\succ'_h$:

$$\succ'_h:\ \{z', y\} \succ \{x, y\} \succ \{x', z\} \succ \{y, z\}... \succ (\textit{the same order with } \succ_h)... \succ \emptyset.$$

[12] Notationally, $C_{-h} = (C_{h'})_{h' \in H \setminus \{h\}}$.

[13] For instance, $C'_{h_1}(\{y,k\}) = \{y,k\}$ and $C'_{h_1}(\{z,y,k\}) = z$. This is the same under C_{h_1} as well.

It is easy to verify that the contracts are BS satisfying both the LAD and the IRC under both C_h and C'_h,[14] and the latter is an improvement over the former for doctor d_1. However, $COP(P, C'_h) = \{z', y\}$, making doctor d_1 worse off.

(iii). Let us now show that without the IRC, the COP does not respect improvements under US and the LAD. Consider only one hospital h and two doctors. The doctors' preferences are as follows:

$$P_{d_1} : \ x, x', x'', \emptyset \text{ and } P_{d_2} : \ y, y', \emptyset.$$

The hospital h's initial choice function C_h is given below:

$C_h(x) = \{x\}$	$C_h(x,y) = \{y\}$	$C_h(y,y') = \{y\}$	$C_h(x',x'',y') = \{y'\}$	
$C_h(x') = \{x'\}$	$C_h(x,y') = \{y'\}$	$C_h(x,x',x'') = \{x\}$	$C_h(x,y,y') = \{y'\}$	
$C_h(x'') = \{x''\}$	$C_h(x',x'') = \{x'\}$	$C_h(x,x',y) = \{y\}$	$C_h(x',y,y') = \{y\}$	$C_h(x,x'',y,y') = \{y'\}$
$C_h(y) = \{y\}$	$C_h(x',y) = \{y\}$	$C_h(x,x',y') = \{y'\}$	$C_h(x'',y,y') = \{y\}$	$C_h(x',x'',y,y') = \{y\}$
$C_h(y') = \{y'\}$	$C_h(x',y') = \{y'\}$	$C_h(x,x'',y) = \{y\}$	$C_h(x,x',x'',y) = \{x,y\}$	$C_h(x,x',x'',y,y') = \{x,y\}$
$C_h(x,x') = \{x'\}$	$C_h(x'',y) = \{y\}$	$C_h(x,x'',y') = \{y'\}$	$C_h(x,x',x'',y') = \{y'\}$	
$C_h(x,x'') = \{x\}$	$C_h(x'',y') = \{y'\}$	$C_h(x',x'',y) = \{y\}$	$C_h(x,x',y,y') = \{y\}$	

The COP outcome at the above instance is $\{x, y\}$. Moreover, it is easy to verify that the contracts are US satisfying the LAD under C_h. However, they do not satisfy the IRC (for instance, $C_h(x, x', y) = \{y\}$ and $C_h(x, x', x'', y) = \{x, y\}$).

[14]They are not US though: $C_h(\{x', z, y\}) = \{x', z\}$ and $C_h(\{x', z, y, z'\}) = \{z', y\}$ (this is the same as C'_h).

Let us now consider the following C'_h, which is an improvement over C_h for doctor d_1.

$C'_h(x) = \{x\}$	$C'_h(x, y) = \{x\}$	$C'_h(y, y') = \{y\}$	$C'_h(x', x'', y') = \{x'\}$	
$C'_h(x') = \{x'\}$	$C'_h(x, y') = \{y'\}$	$C'_h(x, x', x'') = \{x'\}$	$C'_h(x, y, y') = \{y'\}$	
$C'_h(x'') = \{x''\}$	$C'_h(x', x'') = \{x'\}$	$C'_h(x, x', y) = \{x\}$	$C'_h(x', y, y') = \{x'\}$	$C'_h(x, x'', y, y') = \{y'\}$
$C'_h(y) = \{y\}$	$C'_h(x', y) = \{x'\}$	$C'_h(x, x', y') = \{x'\}$	$C'_h(x'', y, y') = \{y\}$	$C'_h(x', x'', y, y') = \{x'\}$
$C'_h(y') = \{y'\}$	$C'_h(x', y') = \{x'\}$	$C'_h(x, x'', y) = \{x\}$	$C'_h(x, x', x'', y) = \{x\}$	$C'_h(x, x', x'', y, y') = \{x\}$
$C'_h(x, x') = \{x\}$	$C'_h(x'', y) = \{y\}$	$C'_h(x, x'', y') = \{y'\}$	$C'_h(x, x', x'', y') = \{x'\}$	
$C'_h(x, x'') = \{x\}$	$C'_h(x'', y') = \{y'\}$	$C'_h(x', x'', y) = \{x'\}$	$C'_h(x, x', y, y') = \{x'\}$	

The COP outcome at the above instance is $\{x'\}$, hence doctor d_1 is worse off. Note that under C'_h, the contracts are US satisfying the LAD. However, they do not satisfy the IRC (for instance, $C'_h(x, x', x'') = \{x'\}$ and $C'_h(x, x', x'', y) = \{x\}$).

Acknowledgment

I am grateful to Associate Editor and two anonymous referees for their through comments and suggestions. I thank Tayfun Sönmez, Bumin Yenmez, İsa Hafalir, Utku Ünver, Ahmet Alkan for their comments. I am especially grateful to Orhan Aygün and Bertan Turhan for their stimulating discussions, suggestions, and comments.

References

Afacan, M. O. (2016): "Graduate Admission with Financial Support," *mimeo.*

Aizerman, M. A. and A. V. Malishevski (1981): "General Theory of Best Variants Choice: Some Aspects," *Automatic Control, IEEE Transactions on*, 26(5), 1030–1040.

Aygün, O. and I. Bo (2014): "College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case," *mimeo.*

Aygün, O. and T. Sönmez (2012): "The Importance of Irrelevance of Rejected Contracts in Matching under Weakened Substitutes Conditions," *mimeo.*

——— (2013): "Matching with Contracts: Comment," *American Economic Review*, 103(5), 2050–2051.

Balinski, M. and T. Sönmez (1999): "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, 84, 73–94.

Chambers, C. P. and M. Yenmez (2014): "Choice and Matching," *mimeo.*

Chun, Y. and W. Thomson (1988): "Monotonicity properties of bargaining solutions when applied to economics," *Mathematical Social Sciences*, 15(1), 11–27.

Crawford, V. P. (1991): "Comparative Statics in Matching Markets," *Journal of Economic Theory*, 54, 389–400.

Crawford, V. P. and E. M. Knoer (1981): "Job Matching with Heterogenous Firms and Workers," *Econometrica*, 49(2), 437–450.

Echenique, F. (2012): "Contracts vs Salaries in Matching," *American Economic Review*, 102(1), 594–601.

Echenique, F. and M. B. Yenmez (2015): "How to Control Controlled School Choice," *American Economic Review*, 105(8), 2679–2694.

Ehlers, L. and B. Klaus (2016): "Object Allocation via Deferred-Acceptance Strategy-Prooness and Comparative Statics," *Games and Economic Behavior*, 97, 128–146.

Gale, D. and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

Gale, D. and M. Sotomayor (1985): "Some Remarks On The Stable Matching Problem," *Discrete Applied Mathematics*, 11, 223–232.

Hatfield, J. W. and F. Kojima (2010): "Substitutes and Stability for Matching with Contracts," *Journal of Economic Theory*, 145, 1704–1723.

Hatfield, J. W. and P. R. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95(4), 913–935.

Hirata, D. and Y. Kasuya (2014): "Cumulative Offer Process is Order-Independent," *Economics Letters*, 124, 37–40.

Kadam, S. V. (2015): "Unilateral substitutability implies substitutable completability in many-to-one matching with contracts," *mimeo*.

Kamada, Y. and F. Kojima (2015): "General Theory of Matching Under Distributional Constraints," *mimeo*.

Kelso, A. S. and J. V. P. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50(6), 1483–1504.

Kojima, F. and M. Manea (2010): "Axioms for Deferred Acceptance," *Econometrica*, 78(2), 633–653.

Kominers, S. D. and T. Sönmez (2016): "Matching with Slot-Specific Priorities: Theory," *Theoretical Economics*, 11(2), 683–710.

Roth, A. E. and M. O. Sotomayor (1990): *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Econometric Society Monographs, Cambridge Univ. Press, Cambridge.

Sönmez, T. (2013): "Bidding for Army Career Specialties: Improving the ROTC Branching Mechanism," *Journal of Political Economy*, 121(1), 186–219.

Sönmez, T. and T. B. Switzer (2013): "Matching with (branch-of-choice) contracts at the United States Military Academy," *Econometrica*, 81(2), 451–488.

Thomson, W. (1983): "The Fair Divison of a Fixed Supply Among a Growing Population," *Mathematics of Operation Research*, 8, 319–326.

Yenmez, M. B. (2015): "College Admissions," *GSIA Working Paper 2014-E24*.