A New Ex-Ante Efficiency Criterion and Implications for the Probabilistic Serial Mechanism*

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Abstract

For probabilistic assignment of objects, when only ordinal preference information is available, we introduce an efficiency criterion based on the following domination relation: a probabilistic assignment dominates another assignment if, whenever the latter assignment is ex-ante efficient at a utility profile consistent with the ordinal preferences, the former assignment is ex-ante efficient too; and there is a utility profile consistent with the ordinal preferences at which the latter assignment is not ex-ante efficient but the former assignment is ex-ante efficient. We provide a simple characterization of this domination relation. We revisit an extensively studied assignment mechanism, the Probabilistic Serial mechanism (Bogomolnaia and Moulin [2]), which always chooses a "fair" assignment. We show that the Probabilistic Serial assignment may be dominated by another fair assignment. We provide an almost full characterization of the preference profiles

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at which the serial assignment is undominated among fair assignments.

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1 Introduction

We study the assignment problem in which n objects are to be allocated among n agents such that each agent receives an object and monetary compensations are not possible. Applications include assigning houses to agents or students to schools. Motivated by fairness concerns, probabilistic assignments (lotteries over sure assignments) have been extensively studied in the literature.

Starting with the seminal study by Hylland and Zeckhauser [7], the vast majority of the literature assumes that each agent derives a utility for being assigned an object, and his ex-ante evaluation of a probabilistic assignment is his expected utility for that probabilistic assignment. In other words, agents are endowed with von-Neumann–Morgenstern (vNM) preferences over probabilistic assignments. In this setup, a natural efficiency requirement for a probabilistic assignment is *ex-ante efficiency*: the probabilistic assignment maximizes the sum of the expected utilities. Obviously, evaluating the *ex-ante efficiency* of a probabilistic assignment requires knowledge of the vNM preferences. However, ordinal allocation mechanisms that elicit only preferences over sure objects have been particularly studied in the literature.[1] When an ordinal mechanism is used, agents are asked to report their preference orderings over objects.[2] Therefore, the efficiency of an assignment has to be evaluated based only on the ordinal preference information. In this ordinal environment, we propose the following efficiency criterion: a probabilistic assignment dominates another assignment in social welfare terms, or *sw-dominates*, if

i. whenever the latter assignment is ex-ante efficient at a utility profile consistent

[1] See Bogomolnaia and Moulin [2] for several justifications for observing ordinal mechanisms in practice.

[2] Part of the literature focuses on strict preferences such that each agent reports a complete, transitive, and anti-symmetric ordering over objects. Unless otherwise noted, we allow for weak preferences that are not necessarily anti-symmetric.

with the ordinal preferences, the former assignment is ex-ante efficient too; and

ii. there is a utility profile consistent with the ordinal preferences at which the latter assignment is not ex-ante efficient but the former assignment is ex-ante efficient.

Although our dominance notion is based on all possible utility representations of the ordinal preferences, it does not require any knowledge of the particular utilities. We call an assignment *sw-efficient* if it is not dominated in social welfare terms.

The common method in the literature to ordinally evaluate the efficiency of a probabilistic assignment is based on first order stochastic dominance. This efficiency notion, introduced by Bogomolnaia and Moulin [2], is called *sd-efficiency*: a probabilistic assignment is sd-efficient if it is not stochastically dominated by any other assignment.[3] McLennan [10] shows that an assignment is sd-efficient if and only if there is a utility profile at which it is ex-ante efficient, which readily implies that any sd-efficient assignment sw-dominates any assignment that is not sd-efficient. Here, we show that sw-domination induces a clean ranking of sd-efficient assignments. This clean ranking enables us to welfare-wise compare the well-known assignment mechanisms such as random priority and probabilistic serial assignment mechanisms, which are incomparable according to sd-domination.

We show that if preferences are strict (no agent is indifferent between two different objects), an sd-efficient assignment π sw-dominates another sd-efficient assignment π' if and only if π has a finer support, i.e. the set of agent-object pairs assigned with positive probability in π is a proper subset of the set of agent-object pairs assigned with positive probability in π'. If preferences are weak (indifference is allowed), we extend the support of an assignment so that it possibly includes an agent-object pair that are not assigned with positive probability, provided there is an "equivalent assignment" that includes the pair in its support. Then, we show that an sd-efficient assignment π sw-dominates another sd-efficient assignment π' if and only if π has a finer extended support. A consequence of these results is that when preferences are strict, the only sw-efficient assignments are the Pareto efficient deterministic assignments; and when preferences are weak, the only undominated assignments are

[3] Bogomolnaia and Moulin [2] refers to sd-efficiency as "ordinal efficiency." Here, we use the terminology of Thomson [11].

the sd-efficient assignments in which each agent is indifferent among the objects that he is assigned with positive probability.

Our analysis shows that each sw-efficient assignment is essentially deterministic. This observation indicates a trade-off between fairness and efficiency, since the main motivation for probabilistic assignments is fairness. Put differently, in a setting where randomization is required to establish fairness, the best policy in terms of social welfare efficiency is to establish fairness with a minimum amount of randomization.

To prove Theorem 1 we use the consequences of a result by McLennan [10] and its constructive proof by Manea [9], which together show that for each sd-efficient assignment, a utility profile consistent with the ordinal preferences can be constructed at which the assignment is ex-ante efficient. Here, we are able to describe the general structure of the set of utility profiles at which a given assignment is ex-ante efficient.[4]

In the second part of the paper, we revisit an extensively studied probabilistic assignment mechanism, namely the Probabilistic Serial (*PS*) mechanism. Bogomolnaia and Moulin [2] introduce the *PS* mechanism and show that it always chooses a fair and sd-efficient assignment.[5] We observe that, without sacrificing fairness, the *PS* mechanism can be improved in sw-efficiency. Given this observation, an important question is "When is it possible to have a fair assignment that sw-dominates the serial assignment?". To answer this question, we consider a directed graph, the configuration of which depends on the given ordinal preference profile. We show that a special connectedness property of this graph plays a critical role in understanding at which preference profiles the serial assignment is sw-efficient among fair assignments.

2 The framework

Let N be a set of n agents and A be a set of n objects. For each $i \in N$, the preference relation of i, which we denote by R_i, is a **weak order** on A, i.e. it is transitive and complete. Let P_i denote the associated strict preference relation and I_i the associated indifference relation. Let $\mathcal{R}_i$ denote the set of all possible preference relations for

[4] For example, it follows from the proof of Theorem 1 that for two sd-efficient probabilistic assignments the set of utility profiles at which each assignment is ex-ante efficient are the same if and only if the agent-object pairs assigned with positive probability in these two assignments are the same.

[5] See Section 4 for the formal definition of fairness.

i, and $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$ denote the set of all possible preference profiles, which we also call the **weak preference domain**. Let $\mathcal{R}_i^S \subset \mathcal{R}_i$ denote the set of all possible strict preference relations for i, i.e. the set of all anti-symmetric preference relations in $\mathcal{R}_i$, and $\mathcal{R}^S \equiv \times_{i \in N} \mathcal{R}_i^S$ denote the set of all possible strict preference profiles, which we also call the **strict preference domain**.

A **deterministic assignment** is a one-to-one function from N to A. A deterministic assignment can be represented by an $n \times n$ matrix with rows indexed by agents and columns indexed by objects, and having entries in $\{0, 1\}$ such that each row and each column has exactly one 1. Such a matrix is called a **permutation matrix**. For each $(i, a) \in N \times A$, having 1 in the (i, a) entry indicates that i is assigned a. A probabilistic assignment (an **assignment** hereafter) is a probability distribution over deterministic assignments. An assignment can be represented by an $n \times n$ matrix having entries in $[0, 1]$ such that the sum of the entries in each row and each column is 1. Such a matrix is called a **doubly stochastic matrix**. For each assignment π, and each pair $(i, a) \in N \times A$, the entry π_{ia}, which we also write as $\pi_i(a)$ or $\pi(i, a)$, indicates the probability that i is assigned to a at π. Since each doubly stochastic matrix can be represented as a convex combination of permutation matrices (Birkhoff [1] and von Neumann [12]), the set of all doubly stochastic matrices is the set of all assignments. Let Π be the set of all doubly stochastic matrices.

We denote the collection of all lotteries over A by $\mathcal{L}(A)$. For each $i \in N$, a von-Neumann–Morgenstern (vNM) utility function u_i is a real valued mapping on A, i.e. $u_i : A \to \mathbb{R}$. For each $i \in N$ with preferences $R_i \in \mathcal{R}_i$, a vNM utility function u_i is **consistent** with R_i if for each pair $(a, b) \in A$, we have $u_i(a) \geq u_i(b)$ if and only if $a \; R_i \; b$. We obtain the corresponding preferences of i over $\mathcal{L}(A)$ by comparing the expected utilities, where the expected utility from $\pi_i \in \mathcal{L}(A)$ is $\sum_{a \in A} \pi_i(a) u_i(a)$.

Next, we define the sd-efficiency of an assignment. The formulation of sd-efficiency is independent of any vNM utility specification consistent with the ordinal preferences. Let $\pi, \pi' \in \Pi$, $i \in N$, and $R \in \mathcal{R}$. We say that π_i stochastically dominates π_i' at R_i, or simply π_i **sd-dominates** π_i' at R_i, if for each $a \in A$,

$$\sum_{b : b R_i a} \pi_i(b) \geq \sum_{b : b R_i a} \pi_i'(b).$$

We say that π stochastically dominates π' at R, or simply π **sd-dominates** π' at R, if $\pi \neq \pi'$ and for each $i \in N$, π_i sd-dominates π'_i at R_i. An assignment $\pi \in \Pi$ is **sd-efficient** at R if no assignment sd-dominates π at R. Let $P^{sd}(R)$ denote the set of sd-efficient assignments at R.

3 SW-domination and a characterization

For each utility profile $u = (u_i(.))_{i\in N}$ and assignment π, the **ex-ante social welfare** at (u, π) is the sum of the expected utilities of the agents, that is:

$$SW(u,\pi) = \sum_{(i,a)\in N\times A} \pi_i(a)u_i(a).$$

An assignment π is **utilitarian efficient** at a utility profile u if it maximizes the social welfare at u, i.e. $\pi \in \arg\max_{\pi'\in\Pi} SW(u,\pi')$.

Let $\pi, \pi' \in \Pi$ and $R \in \mathcal{R}$. An assignment π dominates π' in social welfare terms at R, or simply π **sw-dominates** π' at R if

i. for each utility profile u consistent with R, if π' is ex-ante efficient at u, then π is ex-ante efficient at u too, and

ii. there is a utility profile u consistent with R at which π is ex-ante efficient but π' is not ex-ante efficient.

An assignment π is **sw-efficient** at R if there is no assignment π' that sw-dominates π at R. For each $\pi, \pi' \in \Pi$, π and π' are equivalent in social welfare terms at R, or simply π and π' are **sw-equivalent** at R if for each utility profile u consistent with R, π is ex-ante efficient at u if and only if π' is ex-ante efficient at u. An assignment π weakly dominates π' in social welfare terms at R, or simply π **weakly sw-dominates** π' at R if π sw-dominates π' or π and π' are sw-equivalent at R. An assignment π is **strongly sw-efficient** at R if there is no assignment π' that weakly sw-dominates π at R.

Although the sw-domination notion is based on all possible utility representations of the preferences, it does not require any knowledge of the particular utilities. Hence, knowing ordinal preferences suffice for the comparison. However, since there would be a huge collection of utility profiles consistent with any given ordinal preference profile, this comparison can be computationally burdensome. Therefore, it may not be clear which assignments are sw-efficient. In what follows, we provide a characterization of sw-domination. From the characterization, it follows that sw-efficiency implies sd-efficiency. Moreover, a simple relation among sd-efficient assignments identifies whether one of these assignments sw-dominates the other.

First, we introduce some notation. For each $\pi \in \Pi$, we refer to the collection of pairs $(i, a) \in N \times A$ with $\pi_i(a) > 0$ as the **support** of π, denoted by $Sp(\pi)$. For each $\pi, \pi' \in \Pi$, $Sp(\pi) \subsetneq Sp(\pi')$ means that for each pair $(i, a) \in N \times A$, if $\pi_i(a) > 0$ then $\pi'_i(a) > 0$, and there is a pair $(i, a) \in N \times A$ such that $\pi_i(a) = 0$, but $\pi'_i(a) > 0$.

The support notion will be critical in characterizing sw-domination on the strict preference domain. For a characterization on the weak preference domain, an extension of the support notion will be critical. The following relation on (π, R), denoted by $\sim_{(\pi,R)}$, will be helpful to define the extension. For each $(i, a), (j, b) \in N \times A$, $(i, a) \sim_{(\pi,R)} (j, b)$ if and only if $\pi_i(b) > 0$ and $a \; I_i \; b$.

Note that if $(i, a) \sim_{(\pi,R)} (j, b)$, then for each $k \in N$, $(i, a) \sim_{(\pi,R)} (k, b)$. A **cycle** at $\sim_{(\pi,R)}$ is a sequence of pairs (not necessarily distinct) $(i_1, a_1), (i_2, a_2), \ldots, (i_k, a_k) \in N \times A$ such that $(i_1, a_1) \sim_{(\pi,R)} (i_2, a_2) \sim_{(\pi,R)} \cdots \sim_{(\pi,R)} (i_k, a_k) \sim_{(\pi,R)} (i_1, a_1)$.

Let R be a preference profile and π be an assignment. A pair $(i, a) \in N \times A$ is in the **extended support** of π relative to R, denoted by $(i, a) \in ExtSp(\pi, R)$, if there is a cycle of $\sim_{(\pi,R)}$ that contains (i, a). To get some intuition, imagine that $(i, a) \notin Sp(\pi)$ and we trade a small probability along the cycle such that i_1 gets less of a_2 and more of a_1, i_2 gets less of a_3 and more of a_2, and so on. Note that in the new assignment, each agent's expected utility is the same as before; moreover, (i, a) is now included in the support. In a sense, although (i, a) is not included in the support π, it is included in the support of an equivalent assignment. Also observe that, in order to have $(i, a) \in ExtSp(\pi, R) \setminus Sp(\pi)$, there must be an object $b \in A$ such that $a \; I_i \; b$, $\pi_i(b) > 0$. The following example illustrates the support and the extended support of an assignment.

Example 1. *Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Let $R \in \mathcal{R}$ and $\pi \in \Pi$ be as depicted below. Note, for instance, that agent 1 is indifferent between a and b, he prefers a or b to c, and he is assigned b for sure at assignment π.*

R_1	R_2	R_3
a, b	a	a, c
c	b, c	b

π	a	b	c
1	0	1	0
2	0.4	0	0.6
3	0.6	0	0.4

Figure 1: The extended support of π is $ExtSp(\pi, R) = \{(1, a), (1, b), (2, a), (2, b), (2, c), (3, a), (3, c)\}$.

Note that $Sp(\pi) = \{(1, b), (2, a), (2, c), (3, a), (3, c)\}$. *Observe that* $(1, a) \sim_{(\pi,R)} (2, b) \sim_{(\pi,R)} (3, c) \sim_{(\pi,R)} (1, a)$ *is a cycle of* $\sim_{(\pi,R)}$. *Therefore, we have* $(1, a), (2, b) \in ExtSp(\pi, R)$. *Note that* $(3, b) \notin ExtSp(\pi, R)$ *since there is no* $b' \in A$ *such that* $b \; I_3 \; b'$, $\pi_i(b') > 0$. *Thus,* $ExtSp(\pi, R) = \{(1, a), (1, b), (2, a), (2, b), (2, c), (3, a), (3, c)\}$.

Next, we present a characterization of sw-domination.

Theorem 1. *Let* $\pi, \pi' \in \Pi$ *and* $R \in \mathcal{R}$. *The assignment* π *sw-dominates* π' *at* R *if and only if*

i. $\pi' \notin P^{sd}(R)$ *and* $\pi \in P^{sd}(R)$*, or*

ii. $\pi' \in P^{sd}(R)$ *and* $ExtSp(\pi, R) \subsetneq ExtSp(\pi', R)$.

Proof. See section 6.1.[6] □

As a corollary to Theorem 1, for the strict and the weak preference domains we identify all the assignments that are sw-efficient.

Corollary 1. *i. Let* $R \in \mathcal{R}$. *An sd-efficient assignment* π *is sw-equivalent to another sd-efficient assignment* π' *at* R *if and only if* $ExtSp(\pi) = ExtSp(\pi')$.

ii. Let $R \in \mathcal{R}^S$. *An sd-efficient assignment* π *sw-dominates another sd-efficient assignment* π' *at* R *if and only if* $Sp(\pi) \subsetneq Sp(\pi')$*; and* π *is sw-equivalent to* π' *at* R *if and only if* $Sp(\pi) = Sp(\pi')$.

iii. Let $R \in \mathcal{R}^S$. *An assignment* π *is sw-efficient at* R *if and only if it is a Pareto efficient deterministic assignment at* R.

[6] As for the extension to two-sided markets, one can show that the counterpart of Theorem 1 holds for the marriage problem by using the utility profile construction in Dogan and Yildiz [4] vis à vis the use of Manea's construction for the current result.

iv. Let $R \in \mathcal{R}$. An assignment π is sw-efficient at R if and only if π is sd-efficient at $\mathcal{R}$ and each agent is indifferent between the objects he is assigned with positive probability at π.

Proof. See Section 6.2. □

Since the main motivation for probabilistic assignments is fairness, Corollary 1 indicates a contrast between fairness and efficiency. Think of any setting where randomization is required to establish fairness. It follows from our result that the best policy in terms of social welfare efficiency would be to establish fairness with a minimum amount of randomization. For some problems, the contrast between sw-efficiency and fairness may be extreme, in that the only fair assignments are the least sw-efficient ones from among the sd-efficient assignments. For example, when agents have the same preferences, each one of the two well-known fairness requirements for probabilistic assignments, namely sd-no-envy and equal treatment of equals, pins down a unique assignment: agents share each object equally. Note that this assignment is a least sw-efficient assignment of sd-efficient ones, since it has full support.

4 SW-efficiency of the Probabilistic Serial Mechanism

An assignment mechanism is a function $\varphi : \mathcal{R} \to \Pi$, associating an assignment with each preference profile. On the strict preference domain, a widely studied probabilistic assignment mechanism is the probabilistic serial (PS) mechanism. At each $R \in \mathcal{R}^S$, the *PS* assignment is computed by the following algorithm. Consider each object as an infinitely divisible good with a one unit supply that will be eaten by agents in the time interval $[0, 1]$ through the following steps:

Step 1: Each agent eats from his most preferred object. Agents eat at the same speed. When an object is completely eaten, proceed to the next step.

Steps $s \geq 2$: Each agent eats from his most preferred object from among the ones that have not yet been completely eaten. Agents eat at the same speed. When an object is completely eaten, proceed to the next step.

The algorithm terminates when all the objects are exhausted (or equivalently when each agent has eaten in total exactly one unit of objects), and the probability that an agent receives an object in the *PS* assignment is defined as the amount of the object the agent has eaten. We denote the *PS* assignment at R by $\pi^{ps}(R)$.

Given $R \in \mathcal{R}^S$, $a \in A$, and $t \in [0,1]$, we say that a is **exhausted** at time t in the *PS* algorithm at R if at the end of the step that ends when a is completely eaten, each agent has eaten in total t units of the objects. Note that for each pair $a, b \in A$, if a and b are exhausted at different times in the PS algorithm at R, then for each $i, j \in N$ with $\pi^{ps}(i,a) > 0$ and $\pi^{ps}(j,b) > 0$, we have $\pi^{ps}(i, \{c \in A : c\ R_i\ a\}) \neq \pi^{ps}(j, \{c \in A : c\ R_j\ b\})$.

Bogomolnaia and Moulin [2] show that the *PS* mechanism chooses an *sd-efficient* assignment at each strict preference profile. Another well-known probabilistic assignment mechanism is the *random priority (RP) mechanism*, which draws at random an ordering of the agents from the uniform distribution, then lets them choose successively their best remaining object (the first agent in the ordering is assigned to his best object, the second agent to his best among the remaining objects, and so on). Bogomolnaia and Moulin [2] show that although *RP* mechanism does not always choose an sd-efficient assignment, there are preference profiles at which *RP* and *PS* mechanisms choose different assignments such that neither sd-dominates the other. It follows from Corollary 1 that at each preference profile the *PS* assignment weakly sw-dominates the *RP* assignment. To see this, consider any $R \in \mathcal{R}^S$, if the *RP* assignment is not sd-efficient at R, then the *PS* assignment sw-dominates the *RP* assignment. Suppose that both assignments are sd-efficient at R. Since the *RP* assignment chooses each Pareto efficient deterministic assignment with a positive probability, the *RP* assignment has the largest support among the ex-post efficient assignments. Then, each agent-object pair that is assigned with a positive probability in the *PS* assignment at R is also assigned with a positive probability in the *RP* assignment at R. Hence, the *PS* assignment either sw-dominates or is sw-equivalent to the *RP* assignment at R. More generally, it follows that any ex-post efficient assignment weakly sw-dominates the *RP* assignment at any preference profile.

Besides *sd-efficiency*, the *PS* mechanism also satisfies *sd–envy-freeness* (Bogomolnaia and Moulin [2]), which has been a central fairness requirement in the probabilistic assignment literature: an assignment π is **sd–envy-free** at R if for each pair of agents $i, j \in N$, π_i sd-dominates π_j at R_i. However, it follows from Corollary 1 that the PS mechanism is not sw-efficient, since there are preference profiles at which PS

mechanism does not choose a deterministic assignment. One natural question is the following: Given $R \in \mathcal{R}^S$, is the *PS* assignment *sw-efficient* in the class of *sd-envy-free* assignments at R? Our next example shows that there is a strict preference profile for which there is an sd–envy-free assignment that sw-dominates the *PS* assignment and is not sw-dominated by any other *sd–envy-free* assignment.

Example 2. *Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Consider the following preference profile.*

R_1	R_2	R_3
a	a	b
b	c	c
c	b	a

$\pi^{ps}(R)$	a	b	c
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	0	$\frac{3}{4}$	$\frac{1}{4}$

π	a	b	c
1	$\frac{1}{2}$	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	0	$\frac{1}{2}$	$\frac{1}{2}$

Figure 2: The assignment π, which is sd-envy-free at R, sw-dominates $PS(R)$ at R, since $Sp(\pi) \subsetneq Sp(\pi^{ps}(R))$.

Consider the PS assignment $\pi^{ps}(R)$ and an alternative assignment, namely π, both of which are depicted above. Note that π has a finer support. Then, by Theorem 1, π sw-dominates $\pi^{ps}(R)$. Also, it is easy to check that π is sd–envy-free, and any assignment that has a finer support cannot be sd–envy-free.

Now, when is it possible to have an sd–envy-free assignment that sw-dominates the *PS* assignment? To answer this question, given $R \in \mathcal{R}^S$, we define a directed graph $G(R)$ as follows:

Definition. For each $R \in \mathcal{R}^S$, $G(R)$ is a directed graph where each agent-object pair is a vertex and for each vertex pair $(i, a), (j, b)$, there is an edge from (i, a) to (j, b), denoted by $(i, a) \to (j, b)$, if for each pair of objects $x, y \in A$ such that $x \; R_i \; a$ with $\pi^{ps}(i, x) > 0$ and $b \; P_j \; y$ with $\pi^{ps}(j, y) > 0$, we have $x \; P_j \; y$.

Note that one can identify the configuration of $G(R)$ directly from the preference profile R. By Lemma 4 in Section 6.4 we show that for each $a \in A$, $(i, a) \to (j, a)$ indicates that if we increase the probability that a is assigned to i in the *PS* assignment, then j will envy i. We observe that a special connectedness property of this graph plays a critical role in understanding when the *PS* assignment is strongly sw-efficient among the sd–envy-free assignments. In graph theoretic language, a vertex (i, a) is said to be **connected to** to another vertex (j, b) in $G(R)$ if there is a **path**, a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(i, a) \to v_1 \to v_2 \to \cdots v_k \to (j, b)$. Next, we introduce the connectedness property that will be key for our results.

Definition. Let $R \in \mathcal{R}^S$ and $a \in A$. The graph $G(R)$ is ***a*-connected** if for each $i, j \in N$ such that $\pi^{ps}(R)(i, a) > 0$, (i, a) is connected to (j, a) in $G(R)$. The graph $G(R)$ is **connected** if it is a-connected for each $a \in A$.

As an illustrative example, consider the following two extreme preference profiles: Suppose that in the first profile each agent has the same preference relation, whereas in the second profile each agent top-ranks a distinct alternative. The *PS* assignment allocates each object equally between the agents at the first preference profile, and assigns each agent his top choice with probability one at the second preference profile. In both preference profiles, $G(R)$ is connected since for each $a \in A$ and $i, j \in N$ with $\pi^{ps}(i, a) > 0$, we have $(i, a) \to (j, a)$. More specifically, for the first preference profile, where the preferences are exactly the same, observe that for each $a \in A$, if we restrict $G(R)$ to the vertex set $N \times \{a\}$ we obtain the complete graph. Similarly, for the second preference profile, for each $a \in A$, since there is a single agent $i \in N$ with $\pi(i, a) > 0$, if we restrict $G(R)$ to the vertex set $N \times \{a\}$, then we obtain a star-shaped directed graph. Moreover, clearly at both preference profiles, the *PS* assignment is the unique sd-efficient and sd–envy-free assignment. In fact, we next show that connectedness of $G(R)$ is sufficient for the *PS* assignment to be strongly sw-efficient among the sd–envy-free assignments at R.

Proposition 1. *Let $R \in \mathcal{R}^S$. If $G(R)$ is connected, then the PS assignment is strongly sw-efficient among the sd–envy-free assignments.*

Proof. See Section 6.3. □

Next, we introduce a property which turns out to be critical in understanding when connectedness is necessary for the *PS* assignment to be strongly sw-efficient among the sd–envy-free assignments.

Definition. A preference profile $R \in \mathcal{R}^S$ satisfies **betweenness** if for each pair $a, b \in A$ that are simultaneously exhausted in the PS algorithm at R and for each $i \in N$ with $\pi^{ps}(i, a) > 0$, there exists $c \in A$ such that $\pi^{ps}(i, c) > 0$ and $a \; P_i \; c \; P_i \; b$.

Note that the betweenness of a preference profile can be directly and easily verified by the *PS* assignment. Moreover, it directly follows from the definition of betweenness that for each preference profile $R \in \mathcal{R}^S$, if for each distinct $a, b \in A$, a and b are exhausted at different times in the *PS* algorithm, then R satisfies betweenness.[7] The next result shows that if a preference profile R satisfies betweenness, then connectedness of $G(R)$ is necessary for the *PS* assignment to be strongly sw-efficient among the sd–envy-free assignments.

Proposition 2. *Let $R \in \mathcal{R}^S$ satisfy betweenness. If the PS assignment is strongly sw-efficient among the sd–envy-free assignments, then $G(R)$ is connected.*

Proof. See Section 6.4. □

Next example shows that in the absence of betweenness, although the *PS* assignment is sw-efficient among sd–envy-free assignments, $G(R)$ may not be connected.

Example 3. *Let $N = \{1, 2, 3, 4\}$ and $A = \{a, b, c, d\}$. Consider the following preference profile.*

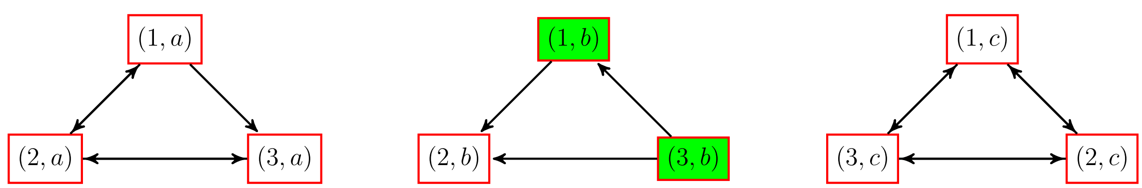
R_1	R_2	R_3	R_4	$\pi^{ps}(R)$	a	b	c	d
a	a	b	c	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$
b	c	c	a	2	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
c	d	d	d	3	0	$\frac{3}{4}$	0	$\frac{1}{4}$
d	b	a	b	4	0	0	$\frac{3}{4}$	$\frac{1}{4}$

First note that objects b and c are exhausted simultaneously at time $3/4$. Since agent 1 ranks c right below b, R violates betweeness. Next, we argue that there is no path that connects the pair $(1, b)$ to $(3, b)$. To see this, first note that only $(4, c)$ is linked to $(3, b)$ and only $(2, c)$ and $(2, a)$ are linked to $(4, c)$. Similarly, note that only $(1, a)$ is linked to $(2, a)$ and $(2, c)$. Since $(1, b)$ is not linked to $(1, a)$, $(2, c)$, $(4, c)$ or $(3, b)$, there is no path that connects $(1, b)$ to $(3, b)$. Finally we argue that $\pi^{ps}(R)$ is the unique assignment that is sd–envy free and sd-efficient at R. To see this, first note that at any sd–envy-free and sd-efficient assignment at R, a should be shared evenly between agents 1 and 2. Given this, to be sd-efficient 2 and 4 should eat from c. Now, for agent 4 not to envy agent

[7] As a special case, whenever each agent-object pair is matched with positive probability, the distinct exhaustion condition is satisfied.

2, 4 should eat $3/4$ of c. Since 1 and 3 rank b over d, 2 and 4 should complete their assignments by equally eating from d. Thus assignments of agents 2 and 4 should be as in $\pi^{ps}(R)$. Next consider the assignment of agent 3. Since a and c are exhausted, 3 can eat from b and d, let p be the amount of b that 3 eats. Now, note that the only value of p that makes 1 and 3 not to envy each other is $3/4$. It follows that $\pi^{ps}(R)$ is the unique sd-efficient among sd–envy-free assignments. Thus we show that in the absence of betweenness, although PS assignment is sw-efficient among sd–envy-free assignments, $G(R)$ may not be connected.

Once we identify when is it possible to sw-dominate the *PS* assignment without sacrificing sd–envy-freeness, the next question is how to obtain such an assignment. The construction in the proof of Proposition 2 implicitly answers this question. Now, we revisit Example 3 to give a rough overview of how can we use this construction to obtain an sd–envy-free assignment that sw-dominates the *PS* assignment. First, consider the preference profile R and $\pi^{ps}(R)$. One can easily check that each object is exhausted at different times in $\pi^{ps}(R)$. Next, consider the graph $G(R)$. Note that if for each $x \in A$, we restrict the $G(R)$ to the vertex set $N \times \{x\}$, we obtain the three graphs below. It directly follows from their configuration that $G(R)$ is a-connected and c-connected. However, $G(R)$ is not b-connected, since $(1, b)$ is not connected to $(3, b)$. To see this, first note that neither $(1, b)$ nor $(2, b)$ is linked to $(3, b)$. Moreover, since only agent 3 top-ranks b and is assigned to c with positive probability, there is no $(i, x) \in N \times \{a, c\}$ with $(i, x) \to (3, b)$.



Now, since $(1, b)$ is not connected to $(3, b)$, we can transfer some amount of b from 3 to 1 without violating sd–envy-freeness. Let us transfer the assignment of b from 3 to 1 until any additional transfer makes 3 to envy 1. This way we can transfer one-quarter the probability of b from 3 to 1. Hence agent 1's assignment is finalized, and we can add the c share of agent 1 in $\pi^{ps}(R)$ to the c assignment of agent 3. Thus, we obtain the assignment π in Example 3, which is sd–envy-free and sw-dominates the *PS* assignment.

Our results in this section are related to a strand of literature that aims to answer at which preference profiles PS assignment is the unique sd–envy-free and sd-efficient assignment.[8] It follows from our Proposition 2 and Corollary 1 that connectedness of $G(R)$ is sufficient for the PS assignment being unique sd–envy-free and sd-efficient assignment among assignments which assign an agent-object pair a positive probability only if the PS assignment assigns a positive probability to that pair. On the other hand, for arbitrary preference profiles, a necessary condition follows from the proof of our Proposition 2 in that if *PS* assignment is the unique sd-efficient and sd–envy-free assignment at a preference profile R, then $G(R)$ must be connected.[9]

5 Conclusion

We propose the notion of social welfare efficiency and show that the previous results can be used to obtain a clean ranking of sd-efficient assignments in terms of this new efficiency notion. This clean ranking enables us to welfare-wise compare the well-known assignment mechanisms such as the random priority and the probabilistic serial assignment mechanisms, which are incomparable according to sd-domination.

Our analysis in the first part of the paper shows that each sw-efficient assignment is essentially deterministic, indicating a trade-off between fairness and efficiency. In the second part of the paper, we focus on the probabilistic serial assignment. We question at which preference profiles the probabilistic serial assignment is sw-efficient among fair assignments. In Propositions 1 and 2, we show that connectedness of a directed graph induced by the preference profile provides an almost full answer to this question.

[8] Heo [6] and Cho [3] provide different sufficient conditions that are not necessary for the uniqueness of the PS assignment.

[9] In the proof of our Proposition 2 for each preference profile R such that $G(R)$ is not connected, we construct an assignment that is sd–envy-free, sd-efficient and different from the *PS* assignment. We use betweenness condition to additionally show that the constructed assignment weakly sd-dominates the *PS* assignment.

6 Appendix

6.1 Proof of Theorem 1

The following lemma, which easily follows from results in Manea [9], plays a central role for the necessity part of our result. Lemma 1 states that for each assignment π that is sd-efficient at R, there is a utility profile u consistent with R such that for each object a and agents i, j,

i. if $(i, a) \in ExtSp(\pi, R)$ and $(j, a) \in ExtSp(\pi, R)$, then $u_i(a) = u_j(a)$

ii. if $(i, a) \in ExtSp(\pi, R)$ and $(j, a) \notin ExtSp(\pi, R)$, then $u_i(a) > u_j(a)$.

In other words, there is a utility profile and a "common" utility function $v : A \to \mathbb{R}$ such that if $(i, a) \in ExtSp(\pi, R)$, then $u_i(a) = v(a)$; and if $(i, a) \notin ExtSp(\pi, R)$, then $u_i(a) < v(a)$.

Lemma 1. *Let $R \in \mathcal{R}$. If $\pi \in \Pi$ is sd-efficient at R, then there exist a utility profile u consistent with R and a function $v : A \to \mathbb{R}$ such that for each $(i, a) \in N \times A$,*

i. if $(i, a) \in ExtSp(\pi, R)$, then $u_i(a) = v(a)$, and

ii. if $(i, a) \notin ExtSp(\pi, R)$, then $u_i(a) < v(a)$.

Proof. Let $R \in \mathcal{R}$. Suppose that $\pi \in \Pi$ is sd-efficient at R. The existence of u and v with the desired properties will easily follow from the way the utility profile is constructed in Manea [9]. For the sake of completeness, we first need to introduce the notation and the results that we need from Manea [9]. Consider the following binary relations on A:

i. $a \rhd b$ iff there is $i \in N$ such that $a \; P_i \; b$ and $\pi_i(b) > 0$.

ii. $a \bowtie b$ iff $a \not\rhd b$ and there is $i \in N$ such that $a \; I_i \; b$ and $\pi_i(b) > 0$.

iii. $a \unrhd b$ iff $a \rhd b$ or $a \bowtie b$.

iv. $a \triangleq b$ iff there is a sequence of objects $a_1, \ldots, a_k$ (with possibly repeated terms) such that $a_1 \bowtie a_2 \bowtie \ldots \bowtie a_k \bowtie a_1$ with $a, b \in \{a_1, \ldots, a_k\}$.[10] Note that $\triangleq$ is an equivalence relation (reflexive, symmetric, transitive). For each $a \in A$, let $[a]$ denote the equivalence class of a.

v. $[a] \gg [b]$ iff $[a] \neq [b]$ and there are $a' \in [a]$, $b' \in [b]$ such that $a' \trianglerighteq b'$ (This relation is defined on the set of equivalence classes of $\triangleq$).

Since $\pi \in \Pi$ is sd-efficient at R, due to Manea [9] there is a utility profile u consistent with R and a function $v : A \to \mathbb{R}$ with the following properties:[11]

i. For each $a \in A$, $v(a)$ is the length of the shortest chain of $\gg$ starting at $[a]$.

ii. For each $i \in A$ such that $\pi_i(a) > 0$, we have $u_i(a) = v(a)$.

iii. For each $i \in A$ such that $\pi_i(a) = 0$ and $\{b \in A | a \ R_i \ b, \pi_i(b) > 0\} = \emptyset$, we have $u_i(a) < \min_{b \in A} v(b)$.

iv. For each $i \in A$ such that $\pi_i(a) = 0$ and $\{b \in A | a \ R_i \ b, \pi_i(b) > 0\} \neq \emptyset$, we have $u_i(a) < \max_{\{b | a R_i b, \pi_i(b) > 0\}} v(b) + 1$.

Now, we are ready to complete the proof. We will show that the utility profile u and the function v satisfying the above four properties also satisfies the two conditions in the statement of the theorem. Clearly, for each $i \in N$ and $a \in A$, $u_i(a) \leq v(a)$. Therefore, we just need to show that $u_i(a) = v(a)$ iff $(i, a) \in ExtSp(\pi, R)$.

Suppose that $(i, a) \in ExtSp(\pi, R)$. Then, there are $(i_1, a_1), \ldots, (i_k, a_k) \in N \times A$ such that $(i, a) \sim_{(\pi,R)} (i_1, a_1) \sim_{(\pi,R)} \cdots \sim_{(\pi,R)} (i_k, a_k) \sim_{(\pi,R)} (i, a)$. Now, since $(i, a) \sim_{(\pi,R)} (i_1, a_1)$, we have $v(a) \geq u_i(a) = v(a_1)$. Similarly, for each $t \in \{2, \ldots, k-1\}$, $v(a_t) \geq u_{i_t}(a_t) = v(a_{t+1})$, and $v(a_k) \geq u_{i_k}(a_k) = v(a)$. Thus, $u_i(a) = v(a)$.

Suppose that $u_i(a) = v(a)$. If $(i, a) \in Sp(\pi)$, then obviously $(i, a) \in ExtSp(\pi, R)$. So, suppose that $(i, a) \notin Sp(\pi)$. Then, $\{b \in A | a \ R_i \ b, \pi_i(b) > 0\} \neq \emptyset$, because otherwise $u_i(a) < \min_{b \in A} v(b) \leq v(a)$. Let $b \in \arg\max_{b' \in \{b \in A | a R_i b, \pi_i(b) > 0\}} v(b')$. Note

[10] Actually, the definition in Manea [9] requires that $a_1 \trianglerighteq a_2 \trianglerighteq \ldots \trianglerighteq a_k \trianglerighteq a_1$ rather than $a_1 \bowtie a_2 \bowtie \ldots \bowtie a_k \bowtie a_1$. Given that π is sd-efficient at R, the two definitions are equivalent (Katta and Sethuraman [8]).

[11] A result by Katta and Sethuraman [8], which characterizes sd-efficient assignments in terms of a property of $\trianglerighteq$, plays an important role in the analysis of Manea [9].

that $u_i(a) < v(b) + 1$ and $a \trianglerighteq b$. Then either $[a] = [b]$ or $[a] \gg [b]$. If $[a] \gg [b]$, then $v(a) \geq v(b) + 1$, contradicting to $v(a) = u_i(a) < v(b) + 1$. So, suppose that $[a] = [b]$. Then, there are $a_1, \ldots, a_k, a_{k+1}, \ldots, a_K$ such that $b \bowtie a_1 \bowtie a_2 \bowtie \ldots \bowtie a_k \bowtie a \bowtie a_{k+1} \bowtie a_{k+2} \bowtie \ldots \bowtie a_K \bowtie b$. Then, there is $i_1 \in N$ such that $b\ I_{i_1}\ a_1$, $\pi_{i_1}(a_1) > 0$; for each $t \in \{1, \ldots, k\}$, there is a_t such that $a_{t-1}\ I_{i_t}\ a_t$, $\pi_{i_t}(a_t) > 0$; and there is i_{k+1} such that $a\ I_{i_{k+1}}\ a_k$. But then $(i_1, b) \sim_{(\pi,R)} (i_2, a_1) \sim_{(\pi,R)} (i_3, a_2) \sim_{(\pi,R)} \cdots \sim_{(\pi,R)} (i_k, a_{k-1}) \sim_{(\pi,R)} (i_{k+1}, a_k) \sim_{(\pi,R)} (i, a) \sim_{(\pi,R)} (i_1, b)$. Thus, $(i, a) \in ExtSp(\pi, R)$. □

The following lemmas will be helpful to present a characterization of sw-domination.

Lemma 2. *Let $R \in \mathcal{R}$. If an assignment π is sd-efficient at R, then there is a utility profile u consistent with R such that the following is true: an assignment π' maximizes $SW(u, \cdot)$ if and only if $Sp(\pi') \subset ExtSp(\pi, R)$.*

Proof. A straightforward corollary to Theorem 1. □

Lemma 3. *Let $R \in \mathcal{R}$. For each assignment π, there is an assignment π^* such that $Sp(\pi^*) = ExtSp(\pi, R)$ and for each utility profile u consistent with R, $SW(u, \pi^*) = SW(u, \pi)$.*

Proof. Take any $(i_0, a_0) \in ExtSp(\pi, R) \backslash Sp(\pi)$. There exists $(i_1, a_1), (i_2, a_2), \ldots, (i_k, a_k) \in N \times A$ such that $(i_0, a_0) \sim_{(\pi,R)} (i_1, a_1) \sim_{(\pi,R)} \ldots \sim_{(\pi,R)} (i_k, a_k) \sim_{(\pi,R)} (i_0, a_0)$. Starting from π, for some small enough $\epsilon > 0$, by transferring ϵ probability of a_t from i_{t-1} to i_t for each $t \in \{1, \ldots, k\}$, and transferring ϵ probability of a_0 from i_k to i_0, we can obtain an assignment π' such that $\pi'_{i_t}(a_t) > 0$ and $\pi'_{i_t}(a_{t+1}) > 0$ for each $t \in \{1, \ldots, k\}$, with the convention that $a_{k+1} = a_0$.

Note that $Sp(\pi') = Sp(\pi) \cup \{(i_0, a_0), (i_1, a_1), (i_2, a_2), \ldots, (i_k, a_k)\}$ and also $Sp(\pi') \subset ExtSp(\pi, R)$. Moreover, each agent receives the same utility at π' and π. Hence, $SW(u, \pi') = SW(u, \pi)$. Once we repeat this procedure for each $(i, a) \in ExtSp(\pi, R) \setminus Sp(\pi)$, we obtain the desired assignment π^*. □

Now, we are ready to prove Theorem 1.

If part: Suppose that $\pi' \notin P^{sd}(R)$ and $\pi \in P^{sd}(R)$. Since $\pi' \notin P^{sd}(R)$, there is no utility profile u consistent with R at which π' is ex-ante efficient. Since $\pi \in P^{sd}(R)$, by Lemma 1 there is a utility profile u consistent with R at which π' is ex-ante efficient, implying that π sw-dominates π'.

Suppose that $\pi' \in P^{sd}(R)$ and $ExtSp(\pi, R) \subsetneq ExtSp(\pi', R)$. By Lemma 2, there is an assignment π^* such that $Sp(\pi^*) = ExtSp(\pi', R)$ and for each utility profile u consistent with R, $SW(u, \pi^*) = SW(u, \pi')$. Now, consider any utility profile u consistent with R at which π^* is ex-ante efficient. Consider a decomposition of π into deterministic assignments, say consisting of $\mu_1, \ldots, \mu_k$. We will argue that there is a decomposition of π^* which includes $\mu_1, \ldots, \mu_k$. Since $Sp(\pi) \subset Sp(\pi^*)$, there is $\epsilon > 0$ such that each entry of $z = \pi^* - \epsilon\pi$ is non-negative. Moreover, each row sum and each column sum of z is $1 - \epsilon$. Then, $\frac{1}{1-\epsilon}z \in \Pi$ and $\frac{1}{1-\epsilon}z$ can be written as a convex combination of deterministic assignments, say $\mu'_1, \ldots, \mu'_t$. Thus, $\pi^* = \epsilon\pi + z$ and therefore π^* can be decomposed into $\mu_1, \ldots, \mu_k, \mu'_1, \ldots, \mu'_t$. Now, since π^* maximizes $SW(u, \cdot)$, the sum of the utilities of the agents at each deterministic assignment in $\{\mu_1, \ldots, \mu_k, \mu'_1, \ldots, \mu'_t\}$ is the same, and it is equal to $SW(u, \pi^*)$. Thus, $SW(u, \pi) = SW(u, \pi^*)$. Hence, whenever π' is ex-ante efficient, π is ex-ante efficient too. Now, since $\pi \in P^{sd}(R)$, by Lemma 1 there is a utility profile u consistent with R and a function $v : A \to \mathbb{R}$ satisfying the properties listed in Lemma 1. Since, $ExtSp(\pi, R) \subsetneq ExtSp(\pi', R)$, there is a pair $(i, a) \in ExtSp(\pi', R) \setminus ExtSp(\pi, R)$. Moreover, $u_i(a) < v(a)$ and therefore $SW(u, \pi) > SW(u, \pi')$. Hence, π sw-dominates π'.

Only if part: Suppose that π sw-dominates π' at R. Then, $\pi \in P^{sd}(R)$. If $\pi' \notin P^{sd}(R)$, then we are done. So suppose that $\pi' \in P^{sd}(R)$. By Lemma 1 there is a utility profile u consistent with R and a function $v : A \to \mathbb{R}$ satisfying the properties listed in Lemma 1 for the assignment π'. In particular, π' is ex-ante efficient at u. Since π sw-dominates π', π is ex-ante efficient at u too. Moreover, by Lemma 3, there is an assignment π^* such that $Sp(\pi^*) = ExtSp(\pi, R)$ and $SW(u, \pi^*) = SW(u, \pi')$. Thus, π^* is ex-ante efficient at u too. By Lemma 2, this is possible only if $Sp(\pi^*) \subset ExtSp(\pi', R)$. Since $\pi' \in P^{sd}(R)$, we have $ExtSp(\pi, R) \subset ExtSp(\pi', R)$. Now, suppose that $ExtSp(\pi, R) = ExtSp(\pi', R)$. Note that by Lemma 3, there is an assignment π^* such that $Sp(\pi^*) = ExtSp(\pi, R)$ and for every utility profile u, $SW(u, \pi^*) = SW(u, \pi)$. Also, there is an assignment π^{**} such that $Sp(\pi^{**}) = ExtSp(\pi', R)$ and for every utility profile u, $SW(u, \pi^{**}) = SW(u, \pi')$. But then, $Sp(\pi^*) = Sp(\pi^{**})$, and since π sw-dominates π', there is a utility profile at which π and π^* are ex-ante efficient but π' and π^{**} are not, which is a contradiction. Hence, $ExtSp(\pi, R) \subsetneq ExtSp(\pi', R)$.

6.2 Proof of Corollary 1

i. Directly follows from the proof of Theorem 1 in section 6.1.

ii. Directly follows from Theorem 1 and the previous item.

iii. Let $R \in \mathcal{R}$ be strict. First, note that each deterministic Pareto efficient assignment at R is sd-efficient at R. Now, for any assignment π that is not deterministic, there is a Pareto efficient deterministic assignment μ such that $Sp(\mu) \subsetneq Sp(\pi)$ (consider an assignment in one of the decompositions of π). These facts, together with Theorem 1, imply Part 2. In particular, an assignment that is not deterministic is sw-dominated by any deterministic assignment in its decomposition.

iv. *Only if part:* Suppose that π is sw-efficient at R. Note that π is sd-efficient. If π is deterministic, then the claim holds. Suppose π is not deterministic. Suppose that there is $i \in N$ and $a, b \in A$ such that $(i, a), (i, b) \in Sp(\pi)$, but $a \ P_i \ b$. Next, consider a decomposition, say consisting of $\{\mu, \mu', \ldots\}$, of π where i is assigned a at μ and b at μ'. Note that μ is sd-efficient. Moreover, $ExtSp(\mu, R) \subset ExtSp(\pi, R)$ because $(i, b) \notin ExtSp(\mu, R)$. Hence, μ sw-dominates π.

   If part: Suppose that π is sd-efficient and each agent is indifferent between the objects he receives with positive probability at π. Take any π' such that $ExtSp(\pi', R) \subsetneq ExtSp(\pi, R)$. Since each agent is indifferent between the objects he receives with positive probability at π or π', at each utility profile consistent with R, the total utility is the same at π and π', implying that π' does not sw-dominate π. Therefore, π is sw-efficient at R, since otherwise it is sw-dominated by an assignment π' with $ExtSp(\pi', R) \subsetneq ExtSp(\pi, R)$ due to Theorem 1.

6.3 Proof of Proposition 1

First, we introduce some notation. For each $R \in \mathcal{R}$, $i \in N$, and $a \in A$, let $U(R_i, a)$ and $L(R_i, a)$ denote the upper and the lower contour sets of R_i at a, that is, $U(R_i, a) = \{b \in A : b \ R_i \ a\}$ and $L(R_i, a) = \{b \in A : a \ R_i \ b\}$. Let P_i stand for the strict part of the preference relation R_i. Let $U(P_i, a)$ and $L(P_i, a)$ denote the strict upper and the strict lower contour sets of R_i at a, that is, $U(P_i, a) = \{b \in A : b \ P_i \ a\}$ and

$L(P_i, a) = \{b \in A : a \ P_i \ b\}$. Let $V = N \times A$ denote the vertex set of $G(R)$ for each $R \in \mathcal{R}$.

Let π be an sd-efficient probabilistic assignment at R. For a contradiction, suppose that π is an sd–envy-free probabilistic assignment that sw-dominates π^{ps} at R, i.e. $Sp(\pi) \subset Sp(\pi^{ps})$. We will show that, if $G(R)$ is connected, then $\pi = \pi^{ps}$, which will yield a contradiction. In showing that, the following result, which is Theorem 1 of Hashimoto et al. [5], will be useful: $\pi = \pi^{ps}$ if and only if for each $a \in A$ and $i, j \in N$ such that $\pi(i, a) > 0$, we have $\pi(j, U(R_j, a)) \geq \pi(i, U(R_i, a))$. We will just show that if $G(R)$ is connected, then for each $a \in A$ and $i, j \in N$ such that $\pi(i, a) > 0$, we have $\pi(j, U(R_j, a)) \geq \pi(i, U(R_i, a))$.

Let $a \in A$. First, we show that for each $(i, a), (j, b) \in V$ such that $(i, a) \to (j, b)$, $\pi(j, U(P_j, b)) \geq \pi(i, U(P_i, a))$. Since $Sp(\pi) \subset Sp(\pi^{ps})$, for each $x \in U(P_i, a)$ and each $y \in L(P_j, b)$ such that $\pi(i, x) > 0$ and $\pi(j, y) > 0$, we have $x \ P_j \ y$. Let z be the most preferred (according to R_j) alternative in $L(P_j, b)$ with $\pi(j, z) > 0$. Since $(i, a) \to (j, b)$, $U(R_i, a) \cap Sp(\pi) \subset U(P_j, z)$. Since π is envy free, $\pi(j, U(P_j, z)) \geq \pi(i, U(P_j, z))$. Hence we obtain $\pi(j, U(R_j, b)) \geq \pi(i, U(R_i, a))$.

Now, since $G(R)$ is a-connected, there is a path that connects (i, a) to (j, a) in $G(R)$. From the above finding, it follows that $\pi(j, U(R_j, a)) \geq \pi(i, U(R_i, a))$.

6.4 Proof of Proposition 2

To prove this result, first we show that for each $a \in A$ and $(i, a), (j, a) \in V$ such that (i, a) is not linked to (j, a) in $G(R)$, in the *PS* assignment we can increase the probability that i receives a without causing j to envy i.

Lemma 4. *For each $a \in A$ and $(i, a), (j, a) \in V$, if $(i, a) \not\to (j, a)$, then there exists $\epsilon_{ij} > 0$ such that $\pi^{ps}(j, U(R_j, a)) > \pi^{ps}(i, U(R_j, a)) + \epsilon_{ij}$.*

Proof. For the proof we need the following two observations.

Observation 1: Since $(i, a) \not\to (j, a)$, there exits $x \in U(R_i, a)$ such that $\pi^{ps}(i, x) > 0$ and at R_j, j ranks x below some $b \in L(R_j, a)$ such that $\pi^{ps}(j, b) > 0$.

Observation 2: For each $y \in U(P_j, a) \setminus U(R_i, a)$ such that $\pi^{ps}(j, y) > 0$, we have $\pi^{ps}(i, y) = 0$. Suppose not. Since $y \in L(P_i, a)$ and $\pi^{ps}(i, y) > 0$, we have $\pi^{ps}(i, U(R_i, y)) > \pi^{ps}(i, U(R_i, a))$. By definition of π^{ps}, we have $\pi^{ps}(i, U(R_i, a)) =$

$\pi^{ps}(j, U(R_j, a))$. Then, $\pi^{ps}(i, U(R_i, y)) > \pi^{ps}(j, U(R_j, y))$, which contradicts the definition of π^{ps}.

It directly follows from Observation 1 and Observation 2 that $\pi^{ps}(i, U(R_i, a)) \geq \pi^{ps}(i, U(R_j, a)) + \pi_i^{ps}(x)$. Now, choose $\epsilon_{ij} = \frac{\pi_i^{ps}(x)}{2}$. Since by definition of π^{ps} we have $\pi^{ps}(j, U(R_j, a)) = \pi^{ps}(i, U(R_i, a))$, we obtain $\pi^{ps}(j, U(R_j, a)) > \pi^{ps}(i, U(R_j, a)) + \epsilon_{ij}$. □

Next we introduce an auxiliary allocation mechanism, which is a generalization of the *PS* mechanism to a setup where the available capacity of an object is not necessarily 1 and can be an arbitrary amount. Let $q \in \mathbb{R}_+^A$ be a quota vector, which specifies, for each object, the available amount of the object. For a given $R \in \mathcal{R}^S$, agents eat starting from their most preferred objects at equal speeds as usual. The algorithm terminates when each object is exhausted (note that an agent may end up eating more than or less than 1 unit of objects). We denote the *PS* assignment at (R, q) by $\pi^{ps}(R, q)$. For each $q \in \mathbb{R}_+^A$, let $Sp(q) = \{a \in A : q_a > 0\}$. As before, $Sp(\pi(R, q))$ denotes the set of agent-object pairs that are assigned with positive probability at $\pi(R, q)$.

Definition. Let $R \in \mathcal{R}^S$ and $q \in \mathbb{R}_+^A$. We say that $Sp(\pi^{ps}(R, \cdot))$ is **upper semi-continuous** at $q \in \mathbb{R}_+^A$ if there exists an $\epsilon > 0$ such that for each $q' \in \mathbb{R}_+^A$ with $||q' - q|| < \epsilon$ and $Sp(q') \subset Sp(q)$, we have $Sp(\pi^{ps}(R, q')) \subset Sp(\pi^{ps}(R, q))$.

Lemma 5. *Let $\vec{1}$ stand for the unit quota vector in which each object has a quota of 1 unit. For a given preference profile $R \in \mathcal{R}^S$, if R satisfies **betweenness**, then $Sp(\pi^{ps}(R, \cdot))$ is **upper semi-continuous** at $\vec{1} \in \mathbb{R}_+^A$.*

Proof. Consider $\pi^{ps}(R)$ (or equivalently $\pi^{ps}(R, \vec{1})$) and for each object $a \in A$, let $t(\pi^{ps}(R), a)$ be the set of objects that are exhausted before a. It follows from the definition of the *PS* mechanism that for each agent i and object a, we have $\pi^{ps}(R)(i, a) > 0$ if and only if $U(P_i, a) \subset t(\pi^{ps}(R), a)$. Note that given the weak order of exhaustion times of the objects while running *PS* at R for different quota vectors q, one can identify the support of $\pi^{ps}(R, q)$. Therefore, given two quota vectors q and q', if the order of exhaustion times of the objects are the same in $\pi^{ps}(R, q)$ and $\pi^{ps}(R, q')$, then $Sp(\pi^{ps}(R, q)) = Sp(\pi^{ps}(R, q'))$.

Now, let ϵ be such that for each $a, b \in A$ that are exhausted at different times in $\pi^{ps}(R)$, $0 < \epsilon < |t_a - t_b|/n$. Note that for each $q' \in \mathbb{R}_+^A$ with $||q' - \vec{1}|| < \epsilon$, none of the exhaustion orders will be reversed while obtaining $\pi^{ps}(R, q')$. That is, if an object a is exhausted before another object b at $\pi^{ps}(R)$, then a is exhausted before b at $\pi^{ps}(R, q')$ too. However, two objects that are exhausted simultaneously at $\pi^{ps}(R)$, may be exhausted at different times at $\pi^{ps}(R, q')$. For the rest, let π stand for $\pi^{ps}(R)$ and π' for $\pi^{ps}(R, q')$.

Now, by contradiction suppose that there exists a pair $(i, a) \in N \times A$ such that $\pi(i, a) = 0$ but $\pi'(i, a) > 0$. It follows that $U(P_i, a) \not\subset t(\pi, a)$ but $U(P_i, a) \subset t(\pi', a)$. We obtain a contradiction by showing that there exists an object $c \in U(P_i, a) \setminus t(\pi', a)$. To see this, first recall that only the objects that are exhausted simultaneously at π may be exhausted at different times at π'. It follows that for each $x \in t(\pi', a) \setminus t(\pi, a)$, x is exhausted at the same time with a at π. From among these let b be the object that i was eating when a is exhausted at π. Since a and b are exhausted simultaneously at π, and $\pi(i, b) > 0$, it follows from betweenness that there exists an object c such that $\pi(i, c) > 0$ and $b \; P_i \; c \; P_i \; a$. Now, since a and b are exhausted simultaneously at π and $b \; P_i \; c$, c is exhausted after a at π. Therefore, by the choice of q', c must be exhausted after a at π'. It follows that although $c \in U(P_i, a)$, $c \notin t(\pi', a)$. $\square$

Lemma 6. *For a given preference profile $R \in \mathcal{R}^S$, let π^t be the partial assignment that is obtained by running PS until time $t \in [0, 1]$ and for each $x \in A$, let $q^t_x = 1 - \pi^t(N, x)$. If $Sp(\pi^{ps}(R, \cdot))$ is upper semi-continuous at $\vec{1}$, then $Sp(\pi^{ps}(R, \cdot))$ is upper semi-continuous at q^t.*

Proof. Let $Sp(\pi^{ps}(R, \cdot))$ be upper semi-continuous at $\vec{1}$. Then, there exists $\epsilon' > 0$ such that for each $q' \in \mathbb{R}^A_+$ with $||q' - q|| < \epsilon'$ and $Sp(q') \subset Sp(q)$, we have $Sp(\pi^{ps}(R, q')) \subset Sp(\pi^{ps}(R, q))$. Let ϵ be such that $0 < \epsilon < \epsilon'$. Note that for each q' with $||q^t - q'|| < \epsilon$ and $Sp(q') \subset Sp(q^t)$, we have $||(\vec{1} - (\vec{1} - q^t + q')|| < \epsilon$. Since $Sp(\pi^{ps}(R))$ is upper semi-continuous at $\vec{1}$, we obtain $Sp(\pi^{ps}(R, \vec{1} - q^t + q')) \subset Sp(\pi^{ps}(R))$. Note that $Sp(\pi^{ps}(R, \vec{1} - q^t + q')) = Sp(\pi^{ps}(R, q')) \cup Sp(\pi^t)$ and $Sp(\pi^{ps}(R)) = Sp(\pi^{ps}(R, q^t)) \cup Sp(\pi^t)$. Next, we argue that $Sp(\pi^{ps}(R, q')) \subset Sp(\pi^{ps}(R, q^t))$. For this conclusion, it is sufficient to show that for each $(i, a) \in Sp(\pi^{ps}(R, q')) \cap Sp(\pi^t)$, we have $(i, a) \in Sp(\pi^{ps}(R, q^t))$. For each $(i, a) \in Sp(\pi^t)$, if $(i, a) \in Sp(\pi^{ps}(R, q'))$, then $q'_a > 0$. Since $Sp(q') \subset Sp(q^t)$, we have $q^t_a > 0$. Thus we have $(i, a) \in Sp(\pi^t)$ and $q^t_a > 0$, note that this is possible only if a is the object that is eaten by agent i at time t. It follows that i first eats a at $\pi^{ps}(R, q^t)$, therefore $(i, a) \in Sp(\pi^{ps}(R, q^t))$. $\square$

Now, we are ready to complete the proof Proposition 2. By contradiction suppose there exists $a \in A$ such that $G(R)$ is not a-connected. First we observe that a cannot be an alternative that is exhausted last in $\pi^{ps}(R)$. By contradiction, suppose there exists $k \in N$ such that $\pi^{ps}(k, U(R_k, a)) = 1$. Now, we argue that $G(R)$ must be a-connected. To see this note that (1) for each $i \in N$ with $\pi^{ps}(i,a) > 0$, a is the last object that is assigned i in $\pi^{ps}(R)$. (2) for each $j \in N$ with $\pi^{ps}(j,a) = 0$, let b be the last alternative that j is assigned in $\pi^{ps}(R)$. Since both a and b are exhausted at last, we have $b \ R_j \ a$. It directly follows from (1) and (2) that for each $(i,a),(j,a) \in V$ such that $\pi(i,a) > 0$, $(i,a) \to (j,a)$. In what follows we will construct an assignment $\pi \neq \pi^{ps}$ that is sd–envy-free, sd–efficient and $Sp(\pi) \subset Sp(\pi^{ps})$.

First let us define a partial assignment $\pi' : N \times A \to [0,1]$ such that for each $i \in N$ and $a \in A$, $\pi'(i,A) \leq 1$ and $\pi'(N,a) \leq 1$. Note that we can still consider sd–envy-freeness of π', and if π' is sd–envy-free then for each $i,j \in N$ we must have $\pi'(i,A) = \pi'(j,A)$. Now let π' be the partial assignment, which is obtained by running the PS algorithm until a is exhausted. That is, if a is exhausted at time $t \leq 1$, then for each $i \in N$ we have (1) for each $b \in A$ such that $\pi^{ps}(i, U(R_i, b)) \leq t$, $\pi'(i,b) = \pi^{ps}(i,b)$, (2) there is at most one object $c \in A$ such that $\pi'(i,c) > 0$ and $\pi^{ps}(i,c) \neq \pi'(i,c)$, (3) $\pi'(i,A) = t$. Note that $\pi'(N,a) = 1$ and for each $b \in A$ that is exhausted after a, $\pi'(N,b) < 1$. Since π' is obtained through running the PS algorithm, all the arguments for the envy-freeness of π^{ps} holds for π'. Thus, we conclude that π' is sd–envy-free. Next, by using the fact that $G(R)$ isn't a-connected, we will construct an sd-envy-free partial assignment π'' via making some small perturbations to π' on the assignment probabilities of the objects that are assigned with positive probability at π' just before time t.

Since $G(R)$ isn't a-connected, there exist $i^*, j^* \in N$ such that there is no path that connects (i^*,a) to (j^*,a). Let I be the set of all $i \in N$ such that there is a path that connects (i^*,a) to (i,a) and let J be the set of all $j \in N$ such that there is a path that connects (j,a) to (j^*,a). Note that since $(i^*,a) \to (i^*,a)$ and $(j^*,a) \to (j^*,a)$, we have $i^* \in I$ and $j^* \in J$. For each $i \in I$ and $j \in J$, since there is no path that connects (i^*,a) to (j^*,a), there cannot be any path that connects (i,a) to (j,a), so $(i,a) \not\to (j,a)$. It follows from Lemma 4 that for each $i,j \in N$ such that $\pi(i,a) > 0$ and $(i,a) \not\to (j,a)$, there exists $\epsilon_{ij} > 0$ such that $\pi^{ps}(j, U(R_j,a)) > \pi^{ps}(i, U(R_j,a)) + \epsilon_{ij}$. Now, for any $\epsilon \leq min_{\{i,j \in N : \pi(i,a) > 0 \text{ and } (i,a) \not\to (j,a)\}} \epsilon_{ij}$, let $2|I|\epsilon_I = 2|J|\epsilon_J = \epsilon$, so we have $\epsilon_I + \epsilon_J \leq \epsilon$.

Next, we define π'' as follows:

i. For each $i \in I$, $\pi''(i,a) = \pi'(i,a) + \epsilon_I$ and for any $b \neq a$, $\pi''(i,b) = \pi'(i,b)$.

ii. For each $j \in J$, $\pi''(j,a) = \pi'(j,a) - \epsilon_J$, let b be the next consumable object for j after a, then $\pi''(j,b) = \epsilon_I + \epsilon_J$ and finally for any c except a and b, $\pi''(j,c) = \pi'(j,c)$.

iii. For each $k \notin I \cup J$ with $\pi(k,a) > 0$, let b be the next consumable object for k after a. Now, let $\pi''(k,b) = \epsilon_I$ and for each $c \neq b$, $\pi''(k,c) = \pi'(k,c)$.

iv. Finally for each $k \in N$ with $\pi(k,a) = 0$, let b be the lowest ranked object that is consumed with positive probability in π'.

   a. If b is exhausted after a, then let $\pi''(k,b) = \pi'(k,b) + \epsilon_I$ and for each $c \neq b$, $\pi''(k,c) = \pi'(k,c)$.

   b. If a and b are exhausted at the same time, then let c be the next object that is consumed by k at π^{ps}. Let $\pi''(k,c) = \pi'(k,c) + \epsilon_I$ and for each $d \neq c$, $\pi''(k,d) = \pi'(k,d)$.

Now, we argue that π'' is envy-free. First, it is easy to see that by our choice of ϵ and Lemma 4, no agent envies another because of a. Second, no agent envies another because of a previously exhausted object, since we kept the probabilities of all such objects as in $\pi^{PS}(R)$, which is envy-free. Finally no agent envies another because of his lowest-ranked object that he is assigned with positive probability, since for each agent the total probability that he is assigned to an object that is at least as good as that object is equal to $t + \epsilon_I$. To see this note that by construction of π'' for each $i \in N$, $\pi''(i,A) = \pi'(i,A) + \epsilon_I = t + \epsilon_I$. Thus, we conclude that π'' is sd–envy-free. Next, note that while $\pi''(N,a) = 1$, for some $b \neq a$ we might have $\pi''(N,b) > 1$. Now, we argue that we can choose ϵ such that for each $b \in A$, $\pi''(N,b) \leq 1$. To see this, first observe that by construction of π'' for each b that is exhausted after a we have $\pi''(N,b) \leq \pi'(N,b) + n \cdot \epsilon$. So, we can choose ϵ so small that $\pi''(N,b) \leq 1$. Hence, we obtain an sd–envy-free partial assignment π'' such that the assignment of a is different from that of $\pi^{ps}(R)$.

Next, we extend the partial assignment π'' to an assignment π^*. Let $q, q'' \in \mathbb{R}^A_+$ be the quota vectors of the objects such that for each $x \in A$, $q_x = 1 - \pi'(N,x)$ and $q''_x = 1-$

$\pi''(N,x)$ respectively. Define the assignment $\pi^* = \pi'' + \pi^{ps}(R, q'')$. First, we argue that $Sp(\pi^*) \subset Sp(\pi^{ps}(R))$. To see this first note that, by the construction of π'', we have $Sp(\pi'') \subset Sp(\pi^{ps}(R))$. Since R satisfies betweenness, it follows from Lemma 5 and 6 that $Sp(\pi^{ps}(R, \cdot))$ is upper semi-continuous at q. Therefore we can choose ϵ so small that $Sp(\pi^{ps}(R, q'')) \subset Sp(\pi^{ps}(R, q))$. Moreover, since at least $\pi^*(i^*, a) \neq \pi^{ps}(i^*, a)$, π^* is different from $\pi^{ps}(R)$. Since we can easily express π^* as an eating mechanism, it is sd–efficient. Finally, we argue that π^* is sd–envy-free. To see this, note that for each $i, j \in N$ and $x \in A$, we have $\pi^*(i, U(R_i, x)) = \pi''(i, U(R_i, x)) + \pi^{ps}(R, q)(i, U(R_i, x))$ and $\pi^*(j, U(R_i, x)) = \pi''(j, U(R_i, x)) + \pi^{ps}(R, q)(j, U(R_i, x))$. Since π'' and $\pi^{ps}(R, q)$ are sd–envy-free, $\pi^*(i, U(R_i, x)) \geq \pi^*(j, U(R_i, x))$. It follows that π^* is sd–envy-free.

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