Ordinal Efficiency and Implementing Random Assignment by Sequential Object Lotteries

Emre Doğan[1]

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Extended Abstract

The assignment problem, in its simplest form, is to distribute n objects to n agents where each agent gets exactly one object. Agents submit strict linear "ordinal" preferences and an assignment rule (whether random or not) will eventually induce a deterministic allocation such that each agent receives one object. Randomization over deterministic allocations is a common tool used to restore ex-ante fairness. However, introducing randomization has tricky consequences on identifying the efficiency notion.

An elegant efficiency concept namely ordinal efficiency is introduced in the seminal paper by (Bogomolnaia and Moulin 2001). There is a considerable amount of important recent works on ordinal efficiency and solutions satisfying this notion, particularly the probabilistic serial rule (PS) introduced in [4] (see for example ([2], [3], [5], [6], [7]). In this work, we give two characterizations of ordinal efficiency. These characterizations first of all improve our understanding of the ordinal efficient assignments, and yield an intuitive procedure to pinpoint assignment matrices representing ordinally efficient assignments. Secondly, they suggest a different implementation of the ordinally efficient assignment mechanisms by sequential object lotteries, contrary to the cardinal implementation method of defining lotteries over deterministic assignment.

We first want to clarify an overlooked difference in two different definitions and/or representations of a random assignment. The canonical definition (but rarely the

[1] National Research University Higher School of Economics, Department of Mathematics. Moscow, Russia. Email: edogan@hse.ru

representation) of a random assignment is a probability distribution over deterministic assignments. Once this distribution is known one can implement this assignment by lotteries over deterministic assignments. The well-known random priority rule (RP) (Abdulkadiroğlu and Sönmez 1998), which is ex-post efficient but not ordinally efficient, naturally defines such a probability distribution. However, even RP solution is not implemented by drawing lotteries from a pool of deterministic assignment. Instead, a uniformly random ordering over the agents is determined first, and then each agent starting from the first in the order picks his best alternative according to his preferences such that another agent preceding him in this order did not choose this object.

The alternative representation (but rarely the definition) of a random assignment is a bistochastic $n \times n$ matrix P where rows stand for agents, columns for objects, and p_{io} is the probability that agent i gets o. PS for example immediately defines a matrix and another tedious procedure is required to find the induced lottery (which is not unique) over deterministic assignment. The non-uniqueness of the induced lottery may result in an unnecessary restriction on the resulting deterministic assignments. Particularly, two lotteries L, L' which induces the same matrix P may have distinct deterministic assignments in its support. Therefore, we use P as a solution to problem and call it the random allocation matrix. The i-th row of P is the random allocation of agent i. Agent i prefers P to Q if P_i first order stochastically dominates Q_i given agent i's preferences, i.e., if for any utility function that represent i's preferences, the expected utility for i from P_i is more than that of i from Q_i. A random allocation P is ordinally efficient if there is no Q where no agent is worse off and at least an agent is strictly better off according to this stochastic dominance relation.

We define an axiom called *sequentially full distribution of objects* (*SFDO*). A top alternative, say o, is fully distributed if the probability that agents who ranked o as the first object adds up to 1. A solution defined by P satisfies SFDO if there is an ordering of objects such that the first object in the order is fully distributed. When we delete the first object from the preference profile, the second object in the order is fully distributed and so on. Theorem 1 states that a random allocation P is ordinally efficient if and only if it satisfies SFDO. This result immediately induces an easy algorithm to check whether an arbitrary P is ordinally efficient or not. Also, at least if the number of alternatives is not big, one can write the information in P on the preference profile and easily check P's efficiency. Moreover, one can define arbitrary random allocations or a rule that systematically calculates random allocations satisfying ordinal efficiency.

Our last conclusion above suggests a new type of implementation of P, namely sequential object lotteries (*SOL*) defined as follows. Given any P, fix an ordering of the objects $o_1, \ldots, o_n$. Run a lottery to determine who gets o_1 where the probability of i receiving o_1 is p_{io_1}. Call s_1 the agent who gets o_1. Now, s_1 leaves the problem with o_1. We update the probabilities at each remaining lottery as follows: For the object lottery o_k, let N_k be the active agents who did not receive an object in the previous lotteries, probability of an active agent i receiving o_k is $p_{io_k}/\sum_{i\in N_k} p_{io_k}$.

Any P can be implemented by a SOL. However, it may be the case that both depending on the fixed ordering of the objects and the structure of P, agent i ranks b above a, receives object a, $p_{ib} > 0$, and i never entered the lottery for b because a preceds b in the fixed order of objects. We say P admits a *procedurally fair* SOL for some ordering of objects if this is never the case for any agent. Theorem 2 states that P is ordinally efficient if and only if it admits a procedurally fair SOL.

JEL classification: C71; D63

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